

Light propagation in 2PN approximation in the monopole and quadrupole field of a body at rest: Initial value problem

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The light trajectory in the gravitational field of one body at rest with monopole and quadrupole structure is determined in the second post-Newtonian (2PN) approximation. The terms in the geodesic equation for light rays are separated into time-independent tensorial coefficients and four kind of time-dependent scalar functions. Accordingly, the first and second integration of geodesic equation can be reduced in each case to only four kind of scalar master integrals. These integrals can be solved in closed form by recurrence relations. The 2PN terms of monopole and quadrupole contribute less than 1 nano-arcsecond to the total light deflection. There are, however, enhanced terms in the 2PN light deflection, both in case of monopole and quadrupole. These enhanced 2PN terms are caused by the use of an impact vector which is indispensable for modeling of real astrometric measurements. In case of grazing light rays at Jupiter and Saturn, the enhanced 2PN terms, caused by the quadrupole structure of the body, amount up to 0.95 micro-arcseconds and 0.29 micro-arcseconds, respectively. Thus, the 2PN quadrupole terms are relevant for high-precision astrometry on the sub-micro-arcsecond scale of accuracy.

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I. INTRODUCTION

Space-based astrometry missions of the European Space Agency (ESA) have initiated a renewed advance of highly precise astrometric measurements. The Hipparcos mission (nominal mission: 1989-1993) [1–3] has reached the milli-arcsecond (mas) level in astrometric precision, while the Gaia mission (nominal mission: 2013-2018) [4–6] has reached the micro-arcsecond (μas) scale of accuracy in angular observations [7–9].

While micro-arcsecond astrometry has been realized both theoretically and technologically within the Gaia mission, the dawning of sub-micro-arcsecond (sub- μas) or even nano-arcsecond (nas) astrometry has come in the strategic focus of astronomers [10, 11]. In particular, ESA has recently called for new science themes to be investigated in terms of feasibility and technological developments, some of which may become candidates for possible astrometry missions within the ESA Science Program in future [12]. For instance, among several others, the M-5 mission Gaia-NIR [13] has been proposed to ESA to operate in the near infrared spectrum and in the μas and sub- μas domain in astrometric accuracy. Recently, in the final recommendations of the ESA Senior Science Committee [14] the Gaia-NIR mission has been selected as one possible candidate for M-sized astrometry mission. Another astrometry mission is the proposed M-5 mission Theia [15] which is operating in the domain of sub- μas in astrometric precision. A further candidate is the space-based mission NEAT (Nearby Earth Astrometric Telescope) [16], originally designed for an astrometric precision of about 50 nas. The proposed NEAT mission consists in adopting a long focal length realized by a formation flight configuration of two satellites. Also feasibility studies of earth-bounded telescopes are under consideration aiming at accuracies up to about 10 nas [17].

The impressive progress of astrometric accuracies necessitates the use of general relativity to model the reduction of observational data. Such relativistic modeling of observational data consists of several aspects and issues which need to be solved [10]: a set of several astronomical reference systems, relativistic description of the process of observations, relativistic equations of motions of Solar System bodies, etc. A scheme of general relativistic modeling of observational data is given by Fig. 1 in [18]. One important aspect thereof concerns the relativistic modeling of propagation of a light signal from the celestial light source (e.g. star, quasar, exoplanet) through the gravitational field of the Solar System towards the observer. The present-day relativistic models, like the Gaia Relativistic Model (GREM) [19–21] and the Relativistic Astrometric Model (RAMOD) [22–26] as used for data reduction of ESA astrometry mission Gaia, are designed for relativistic astrometry at μas -level of accuracy. These models need considerably to be refined for modeling observational data on the sub- μas -level of accuracy. In particular, unprecedented accuracies on the sub- μas -level of accuracy necessitate considerable advancements in the theory of light propagation in the Solar System.

For determining the light trajectory in curved space-time one needs to solve the geodesic equation [19, 27, 28] (e.g. Eqs. (1.2.46) - (1.2.47) in [19])

$$\frac{d^2 x^\alpha(\lambda)}{d\lambda^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx^\nu(\lambda)}{d\lambda} = 0, \quad (1)$$

$$g_{\alpha\beta} \frac{dx^\alpha(\lambda)}{d\lambda} \frac{dx^\beta(\lambda)}{d\lambda} = 0, \quad (2)$$

where $x^\alpha(\lambda)$ are the four-coordinates of the light signal which depend on some affine curve-parameter λ . The solutions of geodesic equation (1) for light rays must satisfy the null-condition (2) as additional constraint along the light trajectory. The Christoffel symbols in (1) are given by

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right), \quad (3)$$

so they are functions of the metric tensor, where $g^{\alpha\beta}$ and $g_{\alpha\beta}$ are the contravariant and covariant components of the metric tensor of space-time; the metric signature is assumed to be $(-, +, +, +)$.

The metric tensor of some Solar System body, which enters the geodesic equation (1) via the Christoffel symbols, cannot be given in its exact form and one has to resort rather than to approximation schemes. Such an approximation scheme is provided by the post-Newtonian expansion of the metric tensor, which is an expansion in terms of inverse powers of the speed of light, given by:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)} + h_{\alpha\beta}^{(3)} + h_{\alpha\beta}^{(4)} + \mathcal{O}(c^{-5}), \quad (4)$$

where $h_{\alpha\beta}^{(2)} = \mathcal{O}(c^{-2})$ and $h_{\alpha\beta}^{(3)} = \mathcal{O}(c^{-3})$ are the 1PN and 1.5PN corrections and $h_{\alpha\beta}^{(4)} = \mathcal{O}(c^{-4})$ is the 2PN correction of the light-ray metric; see also endnote [91]. The justification of such an expansion is based on the fact that the gravitational fields in the Solar System are weak and the velocities of matter are slow,

$$\frac{m}{P} \ll 1 \quad \text{and} \quad \frac{v}{c} \ll 1, \quad (5)$$

where $m = GM/c^2$ is the Schwarzschild radius of some Solar System body with mass M and radius P , and v stands representatively for possible velocities of matter (orbital velocities, rotational motions, inner circulations, oscillations, etc.).

Let the curved space-time assumed to be covered by the four-coordinates $x^\mu = (ct, \mathbf{x})$, where $\mathbf{x}$ are the spatial coordinates and t is the coordinate time. A unique interpretation of astrometric observations requires the determination of light trajectory $\mathbf{x}(t)$ as function of coordinate time. Consider a light signal which is emitted by some light source having four-coordinates $(ct_0, \mathbf{x}_0)$. In the Newtonian theory that signal would travel along some straight line: $\mathbf{x}_N(t) = \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma}$ in some unit-direction $\boldsymbol{\sigma}$. In curved space-time the light trajectory is given by a complicated curvilinear trajectory

$$\mathbf{x}(t) = \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\mathbf{x}(t, t_0), \quad (6)$$

where $\Delta\mathbf{x}(t, t_0)$ are corrections to the unperturbed light ray. More precisely, these statements are valid in case of a weak gravitational field and ordinary topology of space-time. Inserting the post-Newtonian expansion of the metric tensor (4) into the geodesic equation (1) implies a corresponding post-Newtonian expansion of the light trajectory,

$$\mathbf{x}(t) = \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\mathbf{x}_{1\text{PN}}(t, t_0) + \Delta\mathbf{x}_{1.5\text{PN}}(t, t_0) + \Delta\mathbf{x}_{2\text{PN}}(t, t_0) + \mathcal{O}(c^{-5}), \quad (7)$$

where the post-Newtonian corrections $\Delta\mathbf{x}_{1\text{PN}} = \mathcal{O}(c^{-2})$ and $\Delta\mathbf{x}_{1.5\text{PN}} = \mathcal{O}(c^{-3})$ are the 1PN and 1.5PN terms, while the post-post-Newtonian correction $\Delta\mathbf{x}_{2\text{PN}} = \mathcal{O}(c^{-4})$ is the 2PN term of the light ray.

Remarkable advancement in determining the 1PN and 1.5PN terms $\Delta\mathbf{x}_{1\text{PN}}$ and $\Delta\mathbf{x}_{1.5\text{PN}}$ has been achieved during recent decades. These advancements were also triggered by the ESA astrometry missions Hipparcos and Gaia. The problem of light propagation in 1PN and 1.5PN approximation has been solved for the following systems: monopoles at rest [19], monopoles in motion [18, 29, 30], spin-dipoles at rest [29], spin-dipoles in motion [31], quadrupoles at rest [29], quadrupoles in motion [32], full set of mass-multipoles and spin multipoles at rest [33–35], full set of mass-multipoles and spin multipoles in slow motion [36, 37] (the calculations in [30–32] are performed in the first order of the post-Minkowskian scheme, that means they are valid for arbitrary motions, especially also for ultra-relativistic motions of the bodies).

However, much less is known about light propagation in 2PN approximation, that means for the terms $\Delta\mathbf{x}_{2\text{PN}}$ in (7). In fact, the light trajectory in 2PN approximation has only been determined in the gravitational field of spherically symmetric bodies, namely either monopoles at rest or monopoles in slow motion:

- 2PN light trajectory in the field of one monopole at rest [19, 21, 38–43],
- 2PN light trajectory in the field of two point-like bodies of a binary system [44],
- 2PN light trajectory in field of one arbitrarily moving monopole [45–47].

These results are by far not sufficient for astrometry on the sub- μas -level of accuracy [10]. Especially, it is a well-known fact that astrometric accuracies on sub-micro-arcsecond or nano-arcsecond level cannot be achieved without accounting further second post-Newtonian terms in the theory of light propagation [41–43, 48–56]. In particular, real Solar System bodies are not simply monopoles but can be of arbitrary shape. To account for the arbitrary shape of the bodies one utilizes a decomposition of the gravitational field in terms of multipoles, which are certain integrals over the stress-energy tensor of these bodies [28, 57–59]. The first term in such an expansion is the monopole term. After the dipole term, which vanishes if the origin of the spatial coordinates is located at the center of mass of the body, the next term in the multipole-decomposition is the quadrupole term. Accordingly, in this investigation the geodesic equation for light rays in the quadrupole field of one massive body at rest is considered in 2PN approximation and the impact of the quadrupole structure on light deflection is determined.

The manuscript is organized as follows: The metric of one Solar System body being of arbitrary shape and inner structure is presented in Section II and the exact geodesic equation is given in Section III. The unperturbed light ray is described in Section IV. In Section V and VI the light propagation in the gravitational field of one massive body at rest with monopole and quadrupole structure is determined in 1PN and 2PN approximation. The total light deflection is calculated in Section VII. The problem of *enhanced terms* is considered in Section VIII. Finally, a summary and outlook can be found in Section IX. The notation and some details of the calculations are relegated to a set of several appendices.

II. METRIC OF A SOLAR SYSTEM BODY

In this investigation we will consider the propagation of a light signal in the gravitational field generated by one Solar System body at rest. The metric tensor enters the geodesic equation (1) via the Christoffel symbols (3). Hence,

the first step is to consider the metric of a Solar System body. If the gravitational fields are weak, then the so-called post-Minkowskian (PM) expansion (weak-field expansion) of the metric tensor can be utilized, which is an expansion of the metric tensor in powers of the gravitational constant

$$g_{\alpha\beta}(t, \mathbf{x}) = \eta_{\alpha\beta} + \sum_{n=1}^{\infty} G^n h_{\alpha\beta}^{(\text{nPM})}(t, \mathbf{x}), \quad (8)$$

where $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$ is the flat Minkowskian space-time and $h_{\alpha\beta}^{(\text{nPM})}$ are small metric perturbations, which vanish at Minkowskian past infinity $\mathcal{J}_M^-$ (cf. Fig. 34.2 of Section 34 in [28])

$$\lim_{\substack{x \rightarrow \infty \\ t + \frac{x}{c} = \text{const}}} g_{\alpha\beta}(t, \mathbf{x}) = \eta_{\alpha\beta}. \quad (9)$$

In reality the massive Solar System bodies are of arbitrary shape and inner structure and the bodies can be in arbitrarily rotational and oscillating motions. The Multipolar Post-Minkowskian (MPM) approach provides a robust framework to solve Einstein's field equations for such complex gravitational systems. The MPM formalism has originally been introduced in [61] and later considerably been extended in a series of articles [62–67] to determine the metric density, $\bar{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}$, in the exterior of an isolated source of matter, where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor. In the MPM approach the space-time is covered by harmonic coordinates, in line with the IAU (International Astronomical Union) resolutions [60]; harmonic coordinates are Cartesian-like coordinates which are Cartesian at spatial infinity and they are defined by the gauge-condition $\bar{g}^{\alpha\beta}_{, \beta} = 0$ [28, 57–59]. An isolated gravitational system is defined by the Fock-Sommerfeld boundary conditions, which imply flatness of the metric at spatial infinity (9) and the constraint of no-incoming gravitational radiation is imposed at Minkowskian past null infinity [21, 58, 60, 68, 69] (e.g. Eq. (10) in [69]).

The MPM approach was mainly developed for theoretical understanding of the generation of gravitational waves by some isolated source of matter, like inspiralling binary stars which consist of compact objects like black holes or neutron stars. The compact source of matter can of course also be interpreted as some massive Solar System body. Thus, the MPM approach allows us to obtain the metric perturbations in (8).

The MPM formalism utilizes the post-Minkowskian expansion of metric density. The most general solution for the metric density is given in terms of six so-called source-multipoles [70, 71]. Harmonic coordinates are not unique but allow for a residual gauge transformation. Using this freedom in the choice of harmonic coordinates, it has been demonstrated in [61, 62] that the metric density can finally be expressed in terms of only two symmetric trace-free (STF) multipoles: mass-multipoles $\hat{M}_L$ (describing shape, inner structure and oscillations of the body) and spin-multipoles $\hat{S}_L$ (describing rotational motions and inner currents of the body). For an explicit form of these multipoles in the first post-Minkowskian approximation it is referred to Eqs. (5.33) and (5.35) in [67].

The metric density $\bar{g}^{\alpha\beta}$ and the metric tensor $g_{\alpha\beta}$ contain the same physical information, because they are related bijective to each other [28, 57–59]; some relations are also given in Appendix D in [72]. From the metric density $\bar{g}^{\alpha\beta}$ one may deduce the metric tensor $g_{\alpha\beta}$. A detailed description about how to obtain the metric tensor $g_{\alpha\beta}$ in (8) from the metric density $\bar{g}^{\alpha\beta}$ of the MPM formalism can be found in our recent article [72]. In this way one arrives at the following post-Minkowskian expansion of the metric tensor for a Solar System body in the so-called canonical harmonic gauge [61, 62, 64, 72–75] (cf. Eqs. (78) and (79) in [72])

$$g_{\alpha\beta}(t, \mathbf{x}) = \eta_{\alpha\beta} + \sum_{n=1}^{\infty} G^n h_{\alpha\beta}^{(\text{nPM})}(\hat{M}_L(s), \hat{S}_L(s)). \quad (10)$$

In (10) the origin of the spatial axes of the coordinate system can be assumed to be located at the center of mass of the body and $s = t - x/c$ is the retarded time (travel time of a light signal from the field point $(t, \mathbf{x})$ to the center of the coordinate system). For the concrete form of the 1PM term $h_{\alpha\beta}^{(1\text{PM})}$ and 2PM term $h_{\alpha\beta}^{(2\text{PM})}$ we refer to Eqs. (109) - (111) and Eqs. (115) - (117), respectively, in [72].

If the speed of the matter (e.g. oscillations, rotational motions and inner currents of the body) is small compared to the speed of light, then one may utilize the post-Newtonian (PN) expansion (weak-field slow-motion expansion), which is an expansion of the metric tensor in inverse powers of the speed of light, and reads up to terms of the order $\mathcal{O}(c^{-5})$ as follows,

$$g_{\alpha\beta}(t, \mathbf{x}) = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)}(t, \mathbf{x}) + h_{\alpha\beta}^{(3)}(t, \mathbf{x}) + h_{\alpha\beta}^{(4)}(t, \mathbf{x}) + \mathcal{O}(c^{-5}), \quad (11)$$

where $h_{\alpha\beta}^{(n)} = \mathcal{O}(c^{-n})$ are small perturbations to the flat Minkowskian space-time metric. The expansion (11) follows from (8) in case of slow motions. The post-Newtonian expansion (11) is actually a non-analytic series, because at higher

orders $n \geq 8$ non-analytic terms involving powers of logarithms occur [61, 62, 64, 73, 74]. From the MPM formalism it follows that the most general PN solution is given in terms of only two mass-multipoles and spin-multipoles [72]

$$g_{\alpha\beta}(t, \mathbf{x}) = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)}(\hat{M}_L(s)) + h_{\alpha\beta}^{(3)}(\hat{M}_L(s), \hat{S}_L(s)) + h_{\alpha\beta}^{(4)}(\hat{M}_L(s), \hat{S}_L(s)) + \mathcal{O}(c^{-5}), \quad (12)$$

where the metric perturbations $h_{\alpha\beta}^{(2)}$ and $h_{\alpha\beta}^{(3)}$ can be read off from Eqs. (109) - (111) in [72], while $h_{\alpha\beta}^{(4)}$ is given by Eqs. (115) - (117) in [72].

In case of stationary source of matter the multipoles and the metric perturbations are time-independent and then the post-Newtonian expansion of the metric tensor reads

$$g_{\alpha\beta}(\mathbf{x}) = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)}(\hat{M}_L) + h_{\alpha\beta}^{(3)}(\hat{S}_L) + h_{\alpha\beta}^{(4)}(\hat{M}_L) + \mathcal{O}(c^{-5}). \quad (13)$$

The terms $h_{\alpha\beta}^{(2)}$ are given by Eqs. (127) and (129) in [72], the term $h_{\alpha\beta}^{(3)}$ is given by Eqs. (128) in [72], and $h_{\alpha\beta}^{(4)}$ is given by Eqs. (134) - (136) in [72]. The STF multipoles are given by Eqs. (125) and (126) in [72] and read

$$\hat{M}_L = \int d^3x \hat{x}_L \frac{T^{00} + T^{kk}}{c^2} + \mathcal{O}(c^{-5}), \quad (14)$$

$$\hat{S}_L = \text{STF}_L \int d^3x \hat{x}_{L-1} \epsilon_{ijk} x^j \frac{T^{0k}}{c} + \mathcal{O}(c^{-5}), \quad (15)$$

where $T^{\alpha\beta}$ is the stress-energy tensor of the body and the integration runs over the three-dimensional volume of the body; the STF notation is given in Appendix B.

As described in the introductory Section, our knowledge about 2PN effects in the theory of light propagation in the Solar System is restricted to the case of light propagation in the gravitational field generated by bodies with mass-monopole structure. In this investigation we will go beyond the mass-monopole approximation and consider the problem of light propagation in the 2PN approximation in the field of one body at rest with time-independent monopole and quadrupole structure, while higher mass-multipoles are neglected. Furthermore, we do not consider any rotational motions or inner circulations of the massive body. Accordingly, there are no spin-multipoles and, therefore, no 1.5PN terms in the metric tensor. The mass-dipole term vanishes $M_a = 0$ because the origin of the coordinate system is assumed to be at the center of mass of the body. That means for our purposes the metric (13) simplifies to the form

$$g_{\alpha\beta}(\mathbf{x}) = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)}(M, \hat{M}_{ab}) + h_{\alpha\beta}^{(4)}(M, \hat{M}_{ab}) + \mathcal{O}(c^{-6}). \quad (16)$$

There are no terms of the order $\mathcal{O}(c^{-3})$ and $\mathcal{O}(c^{-5})$ in the metric tensor (16) because mass-multipoles are assumed to be time-independent and spin-multipoles are neglected. The metric perturbations in (16) have been obtained in our article [72] and they are given below: the 1PN terms $h_{\alpha\beta}^{(2)}$ are given by Eqs. (41) and (42) and the 2PN terms $h_{\alpha\beta}^{(4)}$ are given by Eqs. (69) and (70). The monopole and the STF quadrupole moment in (16) are given by

$$M = \int d^3x \frac{T^{00} + T^{kk}}{c^2} + \mathcal{O}(c^{-5}), \quad (17)$$

$$\hat{M}_{ab} = \int d^3x \hat{x}_{ab} \frac{T^{00} + T^{kk}}{c^2} + \mathcal{O}(c^{-5}), \quad (18)$$

where $\hat{x}_{ab} = \hat{x}^{ab} = x^a x^b - \delta^{ab} x^2/3$. The integrals in (17) and (18) run over the three-dimensional volume of the body, where the stress-energy tensor of the body is assumed to be time-independent; see also endnote [92]. The explicit mathematical form of the quadrupole moment (18) is given below by Eq. (105) for the model of an axial symmetric body, which is a good approximation for the giant planets of the Solar System; see also endnote [93].

III. THE EXACT GEODESIC EQUATION

The trajectory of a light signal propagating in curved space-time is determined by the geodesic equation (1) and isotropic condition (2), which in terms of coordinate time read as follows [19, 28, 58] (e.g. Eqs. (1.2.48) - (1.2.49) in

[19] or Eqs. (7.20) - (7.23) in [58]):

$$\frac{\ddot{x}^i(t)}{c^2} + \Gamma_{\mu\nu}^i \frac{\dot{x}^\mu(t)}{c} \frac{\dot{x}^\nu(t)}{c} - \Gamma_{\mu\nu}^0 \frac{\dot{x}^\mu(t)}{c} \frac{\dot{x}^\nu(t)}{c} \frac{\dot{x}^i(t)}{c} = 0, \quad (19)$$

$$g_{\alpha\beta} \frac{\dot{x}^\alpha(t)}{c} \frac{\dot{x}^\beta(t)}{c} = 0, \quad (20)$$

where $g_{\alpha\beta}$ are the covariant components of the metric tensor of space-time and a dot denotes total derivative with respect to coordinate time. The isotropic condition (20) states that light trajectories are null rays. This condition is satisfied at any point along the light trajectory; cf. text below Eq. (63) as well as text below Eqs. (85) and (102). The Christoffel symbols in (19) are given by Eq. (3).

The metric and the Christoffel symbols (3) are functions of the entire space-time $(t, \mathbf{x})$, while if one inserts these symbols in the geodesic equation (19) then the metric and the Christoffel symbols become relevant only at the coordinates of the photon $\mathbf{x}(t)$. Consequently, the derivatives of the metric tensor contained in the Christoffel symbols (3) of the geodesic equation (19) must be taken along the light trajectory,

$$\frac{\partial g_{\alpha\beta}}{\partial x^\mu} = \left. \frac{\partial g_{\alpha\beta}(t, \mathbf{x})}{\partial x^\mu} \right|_{\mathbf{x}=\mathbf{x}(t)}, \quad (21)$$

where $\mathbf{x}(t)$ is the exact light trajectory. The equation (21) means that the differentiations of the metric tensor in (3) have to be performed with respect to the space-time coordinates, and afterwards the light trajectory has to be substituted.

The geodesic equation (19) is a differential equation of second order of one variable, t , thus a well-defined initial-value problem (Cauchy problem) implies two initial conditions: the initial unit direction $\boldsymbol{\mu}$ of the light signal and the spatial position of the light source $\mathbf{x}_0$ at the moment of emission of the light signal:

$$\boldsymbol{\mu} = \left. \frac{\dot{\mathbf{x}}(t)}{|\dot{\mathbf{x}}(t)|} \right|_{t=t_0} \quad \text{with} \quad \boldsymbol{\mu} \cdot \boldsymbol{\mu} = 1, \quad (22)$$

$$\mathbf{x}_0 = \left. \mathbf{x}(t) \right|_{t=t_0}. \quad (23)$$

A graphical elucidation of these vectors is given in Figure 1.

A. The first integration of the exact geodesic equation

As stated above, a unique solution of the first integration of geodesic equation (19) requires the initial condition (22) for the direction $\boldsymbol{\mu}$ of the light ray which is emitted by the light source. The initial condition (22) can be replaced by the initial condition of the direction $\boldsymbol{\sigma}$ of the light ray at past infinity [19, 21, 33, 36, 37, 45–47, 58, 78],

$$\boldsymbol{\sigma} = \left. \frac{\dot{\mathbf{x}}(t)}{c} \right|_{t=-\infty} \quad \text{with} \quad \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1, \quad (24)$$

with $\boldsymbol{\sigma}$ being the unit-direction of the light ray at past infinity. For a graphical elucidation of these statements see also Figure 1. There is a one-to-one correspondence between condition (24) and (22). In fact, one may find a unique relation between the tangent vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ (cf. Section 3.2.3 in [19]). The advantage of (24) is that the integration procedure and the expressions become simpler when using (24) instead of (22). Then, the first integration of geodesic equation (19) yields the coordinate velocity of the light signal,

$$\frac{\dot{\mathbf{x}}(t)}{c} = \int_{-\infty}^t dt \frac{\ddot{\mathbf{x}}(t)}{c^2} = \boldsymbol{\sigma} + \frac{\Delta \dot{\mathbf{x}}(t)}{c}, \quad (25)$$

where the integration variable t (roman style) runs from lower limit of integration $-\infty$ to the upper limit of integration t (mathnormal style) and where $\Delta \dot{\mathbf{x}}(t)/c$ denotes the correction to the unit-direction $\boldsymbol{\sigma}$ of the light ray at past infinity, so that

$$\lim_{t \rightarrow -\infty} \frac{\Delta \dot{\mathbf{x}}(t)}{c} = 0. \quad (26)$$

Let us notice that (24) in combination with (9) implies that the null condition (20) is satisfied at past infinity.

B. The second integration of the exact geodesic equation

As stated above, a unique solution of the second integration of geodesic equation (19) requires the initial condition (23) [19, 21, 33, 36, 37, 45–47, 58, 78],

$$\mathbf{x}_0 = \mathbf{x}(t) \Big|_{t=t_0}, \quad (27)$$

with $\mathbf{x}_0$ being the initial position of the light source at the moment of the emission of the light signal. The second integration of geodesic equation (19) yields the trajectory of the light signal,

$$\mathbf{x}(t) = \int_{t_0}^t dt \frac{\dot{\mathbf{x}}(t)}{c} = \mathbf{x}_0 + c(t - t_0) \boldsymbol{\sigma} + \Delta \mathbf{x}(t, t_0), \quad (28)$$

where the integration variable t (roman style) runs from lower limit of integration t_0 to the upper limit of integration t (mathnormal style) and where $\Delta \mathbf{x}(t, t_0)$ denotes the corrections to the unperturbed light trajectory, so that

$$\lim_{t \rightarrow t_0} \Delta \mathbf{x}(t, t_0) = 0. \quad (29)$$

IV. THE UNPERTURBED LIGHT RAY

The geodesic equation for a light signal, Eqs. (19) and (20), will be solved by iteration. In the zeroth approximation, that is the Newtonian limit, the light signal propagates along a straight line. We assume the light signal to be emitted at the position of the light source, $\mathbf{x}_0$, and travels along the unit-vector $\boldsymbol{\sigma}$. Thus, the coordinate velocity of this so-called unperturbed light ray reads

$$\frac{\dot{\mathbf{x}}_N}{c} = \boldsymbol{\sigma}, \quad (30)$$

where the subindex N stands for Newtonian. Then, the spatial position of the unperturbed light ray at coordinate time t is given by

$$\mathbf{x}_N = \mathbf{x}_0 + c(t - t_0) \boldsymbol{\sigma}. \quad (31)$$

Furthermore, we notice the absolute value of the spatial coordinate of the unperturbed light ray,

$$x_N = \sqrt{x_0^2 + 2c(t - t_0)(\boldsymbol{\sigma} \cdot \mathbf{x}_0) + c^2(t - t_0)^2}. \quad (32)$$

The iterative approach is based on the unperturbed light ray, that means the subsequent calculations are finally expressed in terms of (30) and (31). A further important quantity is the impact vector $\mathbf{d}_\sigma$ and its absolute value d_σ of the unperturbed light ray, which are defined as follows (cf. Eq. (55) in [79])

$$\mathbf{d}_\sigma = \boldsymbol{\sigma} \times (\mathbf{x}_0 \times \boldsymbol{\sigma}) = \boldsymbol{\sigma} \times (\mathbf{x}_N \times \boldsymbol{\sigma}) \quad \text{hence} \quad \boldsymbol{\sigma} \cdot \mathbf{d}_\sigma = 0, \quad (33)$$

$$d_\sigma = |\mathbf{d}_\sigma|, \quad (d_\sigma)^2 = x_0^2 - (\boldsymbol{\sigma} \cdot \mathbf{x}_0)^2 = (x_N)^2 - (\boldsymbol{\sigma} \cdot \mathbf{x}_N)^2. \quad (34)$$

The impact parameter d_σ is assumed to be larger or equal (grazing light ray) than the (equatorial) radius P of the massive Solar System body,

$$d_\sigma \geq P. \quad (35)$$

For graphical illustration see Figure 1. From these definitions one obtains the relations

$$c(t - t_0) = (\boldsymbol{\sigma} \cdot \mathbf{x}_N) - (\boldsymbol{\sigma} \cdot \mathbf{x}_0), \quad (36)$$

$$\mathbf{x}_N = \mathbf{d}_\sigma + (\boldsymbol{\sigma} \cdot \mathbf{x}_N) \boldsymbol{\sigma}, \quad (37)$$

$$(\boldsymbol{\sigma} \cdot \mathbf{x}_N)^2 = (x_N)^2 - (d_\sigma)^2, \quad (38)$$

$$(\boldsymbol{\sigma} \cdot \mathbf{x}_N)^4 = (x_N)^4 - 2(x_N)^2(d_\sigma)^2 + (d_\sigma)^4, \quad (39)$$

which are frequently used in the subsequent considerations.

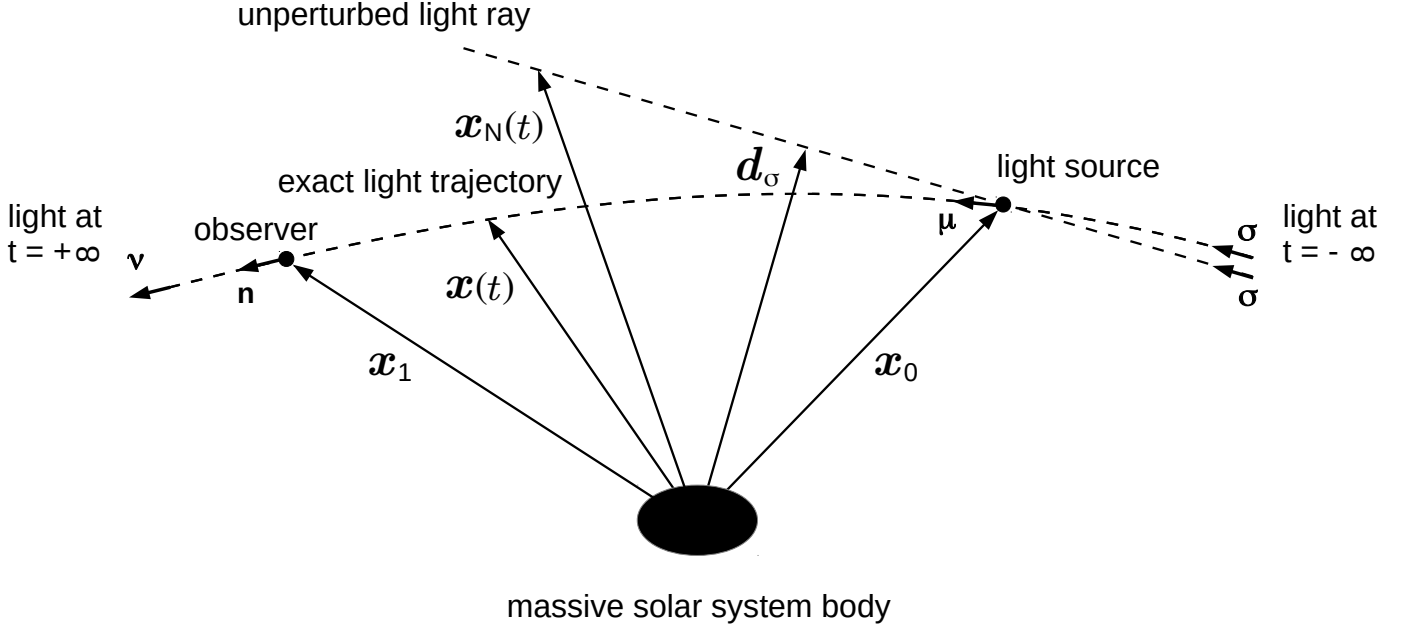


Figure 1: A geometrical representation of the propagation of a light signal through the gravitational field of a massive Solar System body at rest. The origin of the spatial coordinates x^1, x^2, x^3 is assumed to be located at the center of mass of the body. The light signal is emitted by the light source at $\mathbf{x}_0$ in the direction of the unit vector $\boldsymbol{\mu}$ and propagates along the exact light trajectory $\mathbf{x}(t)$ in (28) which is determined by the geodesic equation (19). The unit tangent vector $\boldsymbol{\sigma}$ and $\boldsymbol{\nu}$ of the light trajectory at past infinity and future infinity are given by Eqs. (24) and (95), respectively. The unit tangent vector $\mathbf{n}$ of the light trajectory at the observer's position $\mathbf{x}_1$ is defined by Eq. (117). The unperturbed light ray $\mathbf{x}_N(t)$ is given by Eq. (31) and propagates in the direction of $\boldsymbol{\sigma}$ along a straight line through the position of the light source at $\mathbf{x}_0$. The impact vector $\mathbf{d}_\sigma$ of the unperturbed light ray is given by Eq. (33). The impact vector $\hat{\mathbf{d}}_\sigma$ as given by Eq. (130) is parallel to $\mathbf{d}_\sigma$ and a tiny bit smaller, $\hat{d}_\sigma < d_\sigma$, but is not shown in the diagram.

V. LIGHT PROPAGATION IN 1PN APPROXIMATION

A. The 1PN metric for a body with monopole and quadrupole term

In our investigation, the body is assumed to be at rest and without rotational motion, hence the metric is time-independent. Then, the post-Newtonian series expansion of the metric tensor reads

$$g_{\alpha\beta}(\mathbf{x}) = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)}(\mathbf{x}) + \mathcal{O}(c^{-4}), \quad (40)$$

where $h_{\alpha\beta}^{(2)} = \mathcal{O}(c^{-2})$ and $\mathbf{x}$ is some field point. These metric coefficients are given in terms of so-called symmetric trace-free (STF) mass-multipoles $\hat{M}_L$ and spin-multipoles $\hat{S}_L$ as given by Eqs. (5.33) and (5.35) in [67]. If one takes into account only the mass-monopole and mass-quadrupole terms, then the metric perturbation in 1PN approximation reads [18, 29, 72]

$$h_{00}^{(2)}(\mathbf{x}) = \frac{2G}{c^2} \frac{M}{x} + \frac{3G}{c^2} \frac{\hat{M}_{ab}}{x^5} x^a x^b, \quad (41)$$

$$h_{ij}^{(2)}(\mathbf{x}) = h_{00}^{(2)}(\mathbf{x}) \delta_{ij}, \quad (42)$$

while $h_{0i}^{(2)} = 0$ and the mass-monopole and mass-quadrupole are given by Eqs. (17) and (18), respectively. Let us notice that Eqs. (41) and (42) are identical with Eqs. (1) and (2) in [29] owing to the STF relation (cf. Eq. (1.158) in [57])

$$M_{ab} \hat{x}^{ab} = \hat{M}_{ab} x^{ab} = \hat{M}_{ab} \hat{x}^{ab}, \quad (43)$$

where $M_{ab} = \hat{M}_{ab} + \delta_{ab} M_{kk}/3$. For the same reason Eqs. (41) and (42) agree with Eqs. (145) and (147) in [72].

B. The geodesic equation in 1PN approximation

Inserting the expansion (40) into the geodesic equation (19) yields the geodesic equation in 1PN approximation,

$$\frac{\ddot{x}^i(t)}{c^2} = \frac{1}{2} h_{00,i}^{(2)} - h_{00,j}^{(2)} \frac{\dot{x}^i(t)}{c} \frac{\dot{x}^j(t)}{c} - h_{ij,k}^{(2)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^k(t)}{c} + \frac{1}{2} h_{jk,i}^{(2)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^k(t)}{c} + \mathcal{O}(c^{-4}), \quad (44)$$

where it has been taken into account that the space-time component of the metric perturbation $h_{0i}^{(2)} = 0$ vanishes for a body at rest. In (44) the field point $\mathbf{x}$ as argument of the metric perturbations is taken at the position of the light signal, while $\dot{\mathbf{x}}(t)$ and $\mathbf{x}(t)$ are the coordinate velocity and spatial coordinate of the light signal at this field point. For the geodesic equation in 1PN approximation, one may take the unperturbed light ray ($\dot{\mathbf{x}}_N$ and $\mathbf{x}_N$) on the r.h.s. in (44). In view of (42), the 1PN geodesic equation in (44) then simplifies considerably,

$$\frac{\ddot{x}^i(t)}{c^2} = h_{00,i}^{(2)}(\mathbf{x}_N) - 2 h_{00,j}^{(2)}(\mathbf{x}_N) \sigma^i \sigma^j + \mathcal{O}(c^{-4}). \quad (45)$$

Determining the spatial derivatives of $h_{00}^{(2)}(\mathbf{x})$ in (41) and afterwards replacing $\mathbf{x}$ by $\mathbf{x}_N$ (cf. (Eq. 21)) one gets

$$h_{00,i}^{(2)}(\mathbf{x}_N) = -2 \frac{GM}{c^2} \frac{x_N^i}{(x_N)^3} + 6 \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^5} - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^7}. \quad (46)$$

By inserting (46) into (45) one may separate terms proportional to the monopole and quadrupole, and obtains

$$\frac{\ddot{\mathbf{x}}(t)}{c^2} = \frac{\ddot{\mathbf{x}}_{1PN}^M(t)}{c^2} + \frac{\ddot{\mathbf{x}}_{1PN}^Q(t)}{c^2} + \mathcal{O}(c^{-4}), \quad (47)$$

where the monopole and quadrupole terms are given by

$$\frac{\ddot{\mathbf{x}}_{1PN}^M(t)}{c^2} = \frac{GM}{c^2} \left[-2 \frac{x_N^i}{(x_N)^3} + 4 \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} \right], \quad (48)$$

$$\frac{\ddot{\mathbf{x}}_{1PN}^Q(t)}{c^2} = \frac{G\hat{M}_{ab}}{c^2} \left[6 \delta^{bi} \frac{x_N^a}{(x_N)^5} - 12 \sigma^i \sigma^b \frac{x_N^a}{(x_N)^5} - 15 \frac{x_N^a x_N^b x_N^i}{(x_N)^7} + 30 \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^7} x_N^a x_N^b \right], \quad (49)$$

which agrees with [29]; see also endnote [94]. By means of relations (37) and (38) the r.h.s. of equations (48) and (49) can be rewritten in terms of time-independent tensorial coefficients and time-dependent scalar functions,

$$\frac{\ddot{\mathbf{x}}_{1PN}^M(t)}{c^2} = \frac{GM}{c^2} \left[\mathcal{A}_{(3)}^i \ddot{W}_{(3)}(t) + \mathcal{B}_{(3)}^i \ddot{\chi}_{(3)}(t) \right], \quad (50)$$

$$\frac{\ddot{\mathbf{x}}_{1PN}^Q(t)}{c^2} = \frac{G\hat{M}_{ab}}{c^2} \sum_{n=5,7} \left[\mathcal{C}_{(n)}^{iab} \ddot{W}_{(n)}(t) + \mathcal{D}_{(n)}^{iab} \ddot{\chi}_{(n)}(t) \right], \quad (51)$$

where the time-independent coefficients read

$$\mathcal{A}_{(3)}^i = +2 \sigma^i, \quad (52)$$

$$\mathcal{B}_{(3)}^i = -2 d_\sigma^i, \quad (53)$$

$$\mathcal{C}_{(5)}^{iab} = +6 \sigma^a \delta^{bi} + 3 \sigma^a \sigma^b \sigma^i, \quad (54)$$

$$\mathcal{C}_{(7)}^{iab} = -15 (d_\sigma)^2 \sigma^a \sigma^b \sigma^i + 15 d_\sigma^a d_\sigma^b \sigma^i - 30 \sigma^a d_\sigma^b d_\sigma^i, \quad (55)$$

$$\mathcal{D}_{(5)}^{iab} = +6 d_\sigma^a \delta^{bi} - 15 \sigma^a \sigma^b d_\sigma^i + 18 \sigma^a d_\sigma^b \sigma^i, \quad (56)$$

$$\mathcal{D}_{(7)}^{iab} = -15 d_\sigma^a d_\sigma^b d_\sigma^i + 15 (d_\sigma)^2 \sigma^a \sigma^b d_\sigma^i - 30 (d_\sigma)^2 \sigma^a d_\sigma^b \sigma^i, \quad (57)$$

and the scalar functions are

$$\ddot{W}_{(n)}(t) = \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^n}, \quad (58)$$

$$\ddot{\chi}_{(n)}(t) = \frac{1}{(x_N)^n}. \quad (59)$$

In the following two subsections the first and second integration of differential equations (50) and (51) will be performed.

C. The first integration of geodesic equation in 1PN approximation

The first integration of geodesic equation (47) yields the coordinate velocity of the light signal in 1PN approximation,

$$\frac{\dot{\mathbf{x}}(t)}{c} = \int_{-\infty}^t dt \frac{\ddot{\mathbf{x}}(t)}{c^2} = \boldsymbol{\sigma} + \frac{\Delta \dot{\mathbf{x}}_{\text{1PN}}(t)}{c} + \mathcal{O}(c^{-4}). \quad (60)$$

By inserting (50) and (51) into the r.h.s. of (60) one obtains the first integration of geodesic equation (coordinate velocity) for a light signal in the gravitational field of one body at rest

$$\frac{\Delta \dot{\mathbf{x}}_{\text{1PN}}(t)}{c} = \frac{\Delta \dot{\mathbf{x}}_{\text{1PN}}^{\text{M}}(t)}{c} + \frac{\Delta \dot{\mathbf{x}}_{\text{1PN}}^{\text{Q}}(t)}{c}, \quad (61)$$

where their spatial components are

$$\frac{\Delta \dot{x}_{\text{1PN}}^{\text{M}}(t)}{c} = \frac{GM}{c^2} \left[\mathcal{A}_{(3)}^i \dot{\mathcal{W}}_{(3)}(t) + \mathcal{B}_{(3)}^i \dot{\mathcal{X}}_{(3)}(t) \right], \quad (62)$$

$$\frac{\Delta \dot{x}_{\text{1PN}}^{\text{Q}}(t)}{c} = \frac{G\hat{M}_{ab}}{c^2} \sum_{n=5,7} \left[\mathcal{C}_{(n)}^{iab} \dot{\mathcal{W}}_{(n)}(t) + \mathcal{D}_{(n)}^{iab} \dot{\mathcal{X}}_{(n)}(t) \right]. \quad (63)$$

The scalar master integrals $\dot{\mathcal{W}}_{(n)}$ and $\dot{\mathcal{X}}_{(n)}$ are defined by Eqs. (D1) and (D2), respectively, while their explicit solutions for $n = 3, 5, 7$ are given by Eq. (D6) and Eqs. (D11), (D13), (D15) in Appendix D. One may show that Eq. (62) agrees with [19] and Eq. (63) agrees with [29]; see also endnote [95]. It is easy to check that the solutions in (60) - (63) satisfy the null condition (20) up to terms of the order $\mathcal{O}(c^{-4})$.

D. The second integration of the geodesic equation in 1PN approximation

The second integration of geodesic equation (trajectory) for a light signal in the gravitational field of one body at rest endowed with the mass-monopole and mass-quadrupole in the 1PN approximation reads

$$\mathbf{x}(t) = \int_{t_0}^t dt \frac{\dot{\mathbf{x}}(t)}{c} = \mathbf{x}_0 + (t - t_0) \boldsymbol{\sigma} + \Delta \mathbf{x}_{\text{1PN}}(t, t_0) + \mathcal{O}(c^{-4}). \quad (64)$$

Inserting the r.h.s. of (60) with Eqs. (61) - (63) into (64) yields

$$\Delta \mathbf{x}_{\text{1PN}}(t, t_0) = \Delta \mathbf{x}_{\text{1PN}}^{\text{M}}(t, t_0) + \Delta \mathbf{x}_{\text{1PN}}^{\text{Q}}(t, t_0), \quad (65)$$

where their spatial components are

$$\Delta x_{\text{1PN}}^{\text{M}}(t, t_0) = \frac{GM}{c^2} \left[\mathcal{A}_{(3)}^i \mathcal{W}_{(3)}(t, t_0) + \mathcal{B}_{(3)}^i \mathcal{X}_{(3)}(t, t_0) \right], \quad (66)$$

$$\Delta x_{\text{1PN}}^{\text{Q}}(t, t_0) = \frac{G\hat{M}_{ab}}{c^2} \sum_{n=5,7} \left[\mathcal{C}_{(n)}^{iab} \mathcal{W}_{(n)}(t, t_0) + \mathcal{D}_{(n)}^{iab} \mathcal{X}_{(n)}(t, t_0) \right]. \quad (67)$$

The scalar master integrals $\mathcal{W}_{(n)}$ and $\mathcal{X}_{(n)}$ are defined by Eqs. (D20) and (D21), respectively, while their explicit solutions for $n = 3, 5, 7$ are given by Eqs. (D29), (D31), (D33) and Eqs. (D35), (D37), (D39) in Appendix D. One may show that Eq. (66) agrees with [19] and Eq. (67) agrees with [29]; see also endnote [96].

VI. LIGHT PROPAGATION IN 2PN APPROXIMATION

A. The 2PM metric for a body with monopole and quadrupole term

The post-Newtonian series expansion in 2PM approximation of the metric tensor reads

$$g_{\alpha\beta}(\mathbf{x}) = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)}(\mathbf{x}) + h_{\alpha\beta}^{(4)}(\mathbf{x}) + \mathcal{O}(c^{-6}). \quad (68)$$

The 1PN terms of the metric perturbation, $h_{\alpha\beta}^{(2)}$, were given by Eqs. (41) - (42). The 2PN terms of the metric perturbation, $h_{\alpha\beta}^{(4)}$, in canonical harmonic coordinates read [72]

$$h_{00}^{(4)}(\mathbf{x}) = -\frac{2G^2}{c^4} \frac{M^2}{x^2} - \frac{6G^2}{c^4} \frac{M \hat{M}_{ab}}{x^4} \hat{n}_{ab} - \frac{G^2}{c^4} \frac{\hat{M}_{ab} \hat{M}_{cd}}{x^6} \left(\frac{3}{5} \delta_{ac} \delta_{bd} + \frac{18}{7} \delta_{ac} \hat{n}_{bd} + \frac{9}{2} \hat{n}_{abcd} \right), \quad (69)$$

$$\begin{aligned} h_{ij}^{(4)}(\mathbf{x}) = & \frac{G^2}{c^4} \frac{M^2}{x^2} \left(\frac{4}{3} \delta_{ij} + \hat{n}_{ij} \right) + \frac{G^2}{c^4} \frac{M \hat{M}_{ab}}{x^4} \left(\frac{15}{2} \hat{n}_{ijab} + \frac{32}{7} \delta_{ij} \hat{n}_{ab} - \frac{12}{7} \delta_{a(i} \hat{n}_{j)b} \right) \\ & + \frac{G^2}{c^4} \frac{\hat{M}_{ab} \hat{M}_{cd}}{x^6} \left(\frac{75}{4} \hat{n}_{ijabcd} - \frac{90}{11} \hat{n}_{ijac} \delta_{bd} + \frac{27}{11} \hat{n}_{abcd} \delta_{ij} - \frac{25}{84} \hat{n}_{ij} \delta_{ac} \delta_{bd} + \frac{83}{42} \hat{n}_{ad} \delta_{bc} \delta_{ij} + \frac{16}{35} \delta_{ac} \delta_{bd} \delta_{ij} \right. \\ & \left. + \frac{18}{11} \hat{n}_{acd(i} \delta_{j)b} - \frac{5}{21} \hat{n}_{a(i} \delta_{j)c} \delta_{bd} + \frac{10}{21} \delta_{ci} \delta_{dj} \hat{n}_{ab} - \frac{23}{42} \delta_{b(i} \delta_{j)c} \hat{n}_{ad} - \frac{6}{35} \delta_{ad} \delta_{b(i} \delta_{j)c} \right), \end{aligned} \quad (70)$$

while $h_{0i}^{(4)} = 0$. The STF tensors $\hat{n}_L$ are given by Eq. (B6); for the explicit form of $\hat{n}_{ab}$ and $\hat{n}_{abcd}$ as well as $\hat{n}_{abcdef}$ see Eqs. (B7) and (B8) as well as (B9) in Appendix B. The symmetrization operation is defined by Eq. (B1) in Appendix B and $x = |\mathbf{x}|$ is the absolute of the spatial coordinates of some field point where the origin of the coordinate system is located at the center of mass of the body.

B. The geodesic equation in 2PN approximation

The geodesic equation in 2PN approximation has been given, for instance, in [19] as well as in several articles, e.g. [44–46]. Inserting (68) into the geodesic equation (19) yields

$$\begin{aligned} \frac{\ddot{x}^i(t)}{c^2} = & +\frac{1}{2} h_{00,i}^{(2)} - h_{00,j}^{(2)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^i(t)}{c} - h_{ij,k}^{(2)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^k(t)}{c} + \frac{1}{2} h_{jk,i}^{(2)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^k(t)}{c} - \frac{1}{2} h_{ij}^{(2)} h_{00,j}^{(2)} - h_{00}^{(2)} h_{00,j}^{(2)} \frac{\dot{x}^i(t)}{c} \frac{\dot{x}^j(t)}{c} \\ & + h_{is}^{(2)} h_{sj,k}^{(2)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^k(t)}{c} - \frac{1}{2} h_{is}^{(2)} h_{jk,s}^{(2)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^k(t)}{c} + \frac{1}{2} h_{00,i}^{(4)} - h_{00,j}^{(4)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^i(t)}{c} - h_{ij,k}^{(4)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^k(t)}{c} + \frac{1}{2} h_{jk,i}^{(4)} \frac{\dot{x}^j(t)}{c} \frac{\dot{x}^k(t)}{c} \\ & + \mathcal{O}(c^{-6}). \end{aligned} \quad (71)$$

In (71) the field point $\mathbf{x}$ as argument of the metric perturbations is taken at the position of the light signal, while $\dot{\mathbf{x}}(t)$ and $\mathbf{x}(t)$ are the coordinate velocity and spatial coordinate of the light signal at this field point. For the geodesic equation in 2PN approximation, one has to take the 1PN light ray ($\dot{\mathbf{x}}_{1\text{PN}}$ and $\mathbf{x}_{1\text{PN}}$) for the first four terms on the r.h.s. in (71) and the unperturbed light ray ($\dot{\mathbf{x}}_{\text{N}}$ and $\mathbf{x}_{\text{N}}$) for the other terms on the r.h.s. in (71). Accordingly, in view of (42) and (60), the 2PM geodesic equation in (71) simplifies considerably,

$$\begin{aligned} \frac{\ddot{x}^i(t)}{c^2} = & +h_{00,i}^{(2)}(\mathbf{x}_{1\text{PN}}) - 2h_{00,j}^{(2)}(\mathbf{x}_{1\text{PN}}) \sigma^i \sigma^j + h_{00,i}^{(2)}(\mathbf{x}_{\text{N}}) \frac{\sigma^j \Delta \dot{x}_{1\text{PN}}^j(\mathbf{x}_{\text{N}})}{c} - 2h_{00,j}^{(2)}(\mathbf{x}_{\text{N}}) \frac{\Delta \dot{x}_{1\text{PN}}^j(\mathbf{x}_{\text{N}})}{c} \sigma^i \\ & - 2h_{00,j}^{(2)}(\mathbf{x}_{\text{N}}) \frac{\Delta \dot{x}_{1\text{PN}}^j(\mathbf{x}_{\text{N}})}{c} \sigma^j - h_{00}^{(2)}(\mathbf{x}_{\text{N}}) h_{00,i}^{(2)}(\mathbf{x}_{\text{N}}) + \frac{1}{2} h_{00,i}^{(4)}(\mathbf{x}_{\text{N}}) - h_{00,j}^{(4)}(\mathbf{x}_{\text{N}}) \sigma^i \sigma^j - h_{ij,k}^{(4)}(\mathbf{x}_{\text{N}}) \sigma^j \sigma^k \\ & + \frac{1}{2} h_{jk,i}^{(4)}(\mathbf{x}_{\text{N}}) \sigma^j \sigma^k + \mathcal{O}(c^{-6}), \end{aligned} \quad (72)$$

whereby the spatial derivatives are to be understood that one has to perform first these derivatives and afterwards one has to insert the light ray in the given approximation (cf. Eq. (21)). This equation consists of 10 individual terms,

$$\frac{\ddot{x}}{c^2} = \frac{\ddot{x}_A}{c^2} + \frac{\ddot{x}_B}{c^2} + \frac{\ddot{x}_C}{c^2} + \frac{\ddot{x}_D}{c^2} + \frac{\ddot{x}_E}{c^2} + \frac{\ddot{x}_F}{c^2} + \frac{\ddot{x}_G}{c^2} + \frac{\ddot{x}_H}{c^2} + \frac{\ddot{x}_I}{c^2} + \frac{\ddot{x}_J}{c^2} + \mathcal{O}(c^{-6}), \quad (73)$$

as defined by Eqs. (E2) - (E11). These individual terms are given by Eqs. (E16), (E18), (E19), (E20), (E21), (E22), (E23), (E24), (E25), (E26) in Appendix E. Similar to (47), the sum of these terms in (73) can be separated into monopole and quadrupole and mixed monopole-quadrupole terms,

$$\frac{\ddot{\mathbf{x}}(t)}{c^2} = \frac{\ddot{\mathbf{x}}_{1\text{PN}}^{\text{M}}(t)}{c^2} + \frac{\ddot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}(t)}{c^2} + \frac{\ddot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{M}}(t)}{c^2} + \frac{\ddot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{Q}}(t)}{c^2} + \frac{\ddot{\mathbf{x}}_{2\text{PN}}^{\text{Q}\times\text{Q}}(t)}{c^2} + \mathcal{O}(c^{-6}). \quad (74)$$

The 1PN terms $\ddot{\mathbf{x}}_{1\text{PN}}^{\text{M}}$ and $\ddot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}$ in (74) are proportional to G^1 . These 1PN terms were given by Eq. (47) with (48) and (49), and have already been considered in detail in the previous Section V. The 2PN terms $\ddot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{M}}$ and $\ddot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{Q}}$ and $\ddot{\mathbf{x}}_{2\text{PN}}^{\text{Q}\times\text{Q}}$ in (74) are proportional to G^2 . These 2PN terms will be considered in the following.

1. The terms $\ddot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{M}}$

The terms proportional to $M \times M$ in Eq. (E16) and Eqs. (E18) - (E26) in Appendix E were summarized by Eq. (E27). By inserting $\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}$ and $\Delta\mathbf{x}_{1\text{PN}}^{\text{M}}$ as given by Eqs. (62) and (66), respectively, and using relations (37) - (39) the terms in Eq. (E27) can be written in terms of time-independent tensorial coefficients and time-dependent scalar functions,

$$\frac{\ddot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{M}}(t)}{c^2} = \frac{G^2 M^2}{c^4} \left[\sum_{n=3}^6 \mathcal{E}_{(n)}^i \ddot{\mathcal{W}}_{(n)}(t) + \sum_{n=2}^6 \mathcal{F}_{(n)}^i \ddot{\mathcal{X}}_{(n)}(t) + \mathcal{G}_{(5)}^i \ddot{\mathcal{Y}}_{(5)}(t) + \sum_{n=3,5} \mathcal{H}_{(n)}^i \ddot{\mathcal{Z}}_{(n)}(t) \right]. \quad (75)$$

The step from (E27) to (75) necessitates a similar procedure as the step from Eqs. (48) and (49) to Eqs. (50) and (51). The coefficients $\mathcal{E}_{(n)}^i$, $\mathcal{F}_{(n)}^i$, $\mathcal{G}_{(5)}^i$, $\mathcal{H}_{(n)}^i$ are given by Eqs. (E28) - (E39) in Appendix E. The scalar functions $\ddot{\mathcal{W}}_{(n)}$ and $\ddot{\mathcal{X}}_{(n)}$ were given by Eqs. (58) and (59), respectively, while the scalar functions $\ddot{\mathcal{Y}}_{(n)}$ and $\ddot{\mathcal{Z}}_{(n)}$ read

$$\ddot{\mathcal{Y}}_{(n)}(t) = \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_{\text{N}}}{(x_{\text{N}})^n} \ln \frac{x_{\text{N}} - \boldsymbol{\sigma} \cdot \mathbf{x}_{\text{N}}}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}, \quad (76)$$

$$\ddot{\mathcal{Z}}_{(n)}(t) = \frac{1}{(x_{\text{N}})^n} \ln \frac{x_{\text{N}} - \boldsymbol{\sigma} \cdot \mathbf{x}_{\text{N}}}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}. \quad (77)$$

Actually, in (75) one needs the scalar function $\ddot{\mathcal{Y}}_{(n)}$ only for $n = 5$, while higher values of n are necessary for the mass-quadrupole terms which will be considered in (78).

2. The terms $\ddot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{Q}}$

The terms proportional to $M \times \hat{M}_{ab}$ in Eq. (E16) and Eqs. (E18) - (E26) in Appendix E were summarized by Eq. (E40). By inserting $\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}$ and $\Delta\mathbf{x}_{1\text{PN}}^{\text{M}}$ as given by Eqs. (62) and (66), respectively, as well as $\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}$ and $\Delta\mathbf{x}_{1\text{PN}}^{\text{Q}}$ as given by Eqs. (63) and (67), and using relations (37) - (39), the terms in Eq. (E40) can be written in terms of time-independent tensorial coefficients and time-dependent scalar functions,

$$\frac{\ddot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{Q}}(t)}{c^2} = \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \left[\sum_{n=3}^{10} \mathcal{K}_{(n)}^{iab} \ddot{\mathcal{W}}_{(n)}(t) + \sum_{n=2}^{10} \mathcal{L}_{(n)}^{iab} \ddot{\mathcal{X}}_{(n)}(t) + \sum_{n=7}^9 \mathcal{M}_{(n)}^{iab} \ddot{\mathcal{Y}}_{(n)}(t) + \sum_{n=5}^9 \mathcal{N}_{(n)}^{iab} \ddot{\mathcal{Z}}_{(n)}(t) \right]. \quad (78)$$

The step from (E40) to (78) is achieved by a similar procedure as the step from Eqs. (48) and (49) to Eqs. (50) and (51), but the algebraic effort is more complex. The coefficients $\mathcal{K}_{(n)}^{iab}$, $\mathcal{L}_{(n)}^{iab}$, $\mathcal{M}_{(n)}^{iab}$, $\mathcal{N}_{(n)}^{iab}$ are given by Eqs. (E41) - (E65) in Appendix E. The scalar functions $\ddot{\mathcal{W}}_{(n)}$ and $\ddot{\mathcal{X}}_{(n)}$ were given by Eqs. (58) and (59), while the scalar functions $\ddot{\mathcal{Y}}_{(n)}$ and $\ddot{\mathcal{Z}}_{(n)}$ were given by Eqs. (76) and (77).

3. The terms $\ddot{\mathbf{x}}_{2\text{PN}}^{\text{Q}\times\text{Q}}$

The terms proportional to $\hat{M}_{ab} \times \hat{M}_{cd}$ in Eq. (E16) and Eqs. (E18) - (E26) in Appendix E were summarized by Eq. (E66). By inserting $\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}$ and $\Delta\mathbf{x}_{1\text{PN}}^{\text{M}}$ as given by Eqs. (62) and (66), respectively, as well as $\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}$ and $\Delta\mathbf{x}_{1\text{PN}}^{\text{Q}}$ as given by Eqs. (63) and (67), and using relations (37) - (39), the terms in Eq. (E66) can be written in terms of time-independent tensorial coefficients and time-dependent scalar functions,

$$\frac{\ddot{\mathbf{x}}_{2\text{PN}}^{\text{Q}\times\text{Q}}(t)}{c^2} = \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \left[\sum_{n=5}^{14} \mathcal{P}_{(n)}^{iabcd} \ddot{\mathcal{W}}_{(n)}(t) + \sum_{n=4}^{14} \mathcal{Q}_{(n)}^{iabcd} \ddot{\mathcal{X}}_{(n)}(t) \right]. \quad (79)$$

The step from (E66) to (79) requires a similar procedure as the step from Eqs. (48) and (49) to Eqs. (50) and (51), but the algebraic effort is more complex. The coefficients $\mathcal{P}_{(n)}^{iabcd}$ and $\mathcal{Q}_{(n)}^{iabcd}$ are given by Eqs. (E67) - (E87) in Appendix E. The scalar functions $\ddot{\mathcal{W}}_{(n)}$ and $\ddot{\mathcal{X}}_{(n)}$ were given by Eqs. (58) and (59).

C. The first integration of geodesic equation in 2PN approximation

The first integration of geodesic equation (73) yields the coordinate velocity of the light signal in 2PN approximation,

$$\begin{aligned}\frac{\dot{\mathbf{x}}(t)}{c} &= \int_{-\infty}^t d\tau \frac{\ddot{\mathbf{x}}(\tau)}{c^2} \\ &= \boldsymbol{\sigma} + \frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}(t)}{c} + \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}(t)}{c} + \mathcal{O}(c^{-6}).\end{aligned}\quad (80)$$

By inserting (74) into the r.h.s. of (80) one gets

$$\frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}(t)}{c} = \frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}(t)}{c} + \frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}(t)}{c}, \quad (81)$$

$$\frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}(t)}{c} = \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{M}}(t)}{c} + \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{Q}}(t)}{c} + \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{\text{Q}\times\text{Q}}(t)}{c}. \quad (82)$$

The 1PN terms $\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}$ and $\Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}$ have been determined by Eqs. (62) and (63), respectively, and will not be represented here again.

The first integration of the 2PN monopole-monopole term in (75) is given by

$$\frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{M}}(t)}{c} = \frac{G^2 M^2}{c^4} \left[\sum_{n=3}^6 \mathcal{E}_{(n)}^i \dot{\mathcal{W}}_{(n)}(t) + \sum_{n=2}^6 \mathcal{F}_{(n)}^i \dot{\mathcal{X}}_{(n)}(t) + \mathcal{G}_{(5)}^i \dot{\mathcal{Y}}_{(5)}(t) + \sum_{n=3,5} \mathcal{H}_{(n)}^i \dot{\mathcal{Z}}_{(n)}(t) \right]. \quad (83)$$

The first integration of the 2PN monopole-quadrupole term in (78) is given by

$$\frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{Q}}(t)}{c} = \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \left[\sum_{n=3}^{10} \mathcal{K}_{(n)}^{iab} \dot{\mathcal{W}}_{(n)}(t) + \sum_{n=2}^{10} \mathcal{L}_{(n)}^{iab} \dot{\mathcal{X}}_{(n)}(t) + \sum_{n=7}^9 \mathcal{M}_{(n)}^{iab} \dot{\mathcal{Y}}_{(n)}(t) + \sum_{n=5}^9 \mathcal{N}_{(n)}^{iab} \dot{\mathcal{Z}}_{(n)}(t) \right]. \quad (84)$$

The first integration of the 2PN quadrupole-quadrupole term in (79) is given by

$$\frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}^{\text{Q}\times\text{Q}}(t)}{c} = \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \left[\sum_{n=5}^{14} \mathcal{P}_{(n)}^{abcd} \dot{\mathcal{W}}_{(n)}(t) + \sum_{n=4}^{14} \mathcal{Q}_{(n)}^{abcd} \dot{\mathcal{X}}_{(n)}(t) \right]. \quad (85)$$

The master integrals $\dot{\mathcal{W}}_{(n)}$, $\dot{\mathcal{X}}_{(n)}$, $\dot{\mathcal{Y}}_{(n)}$, $\dot{\mathcal{Z}}_{(n)}$ are defined by Eqs. (D1) - (D4). The solution for the master integral $\dot{\mathcal{W}}_{(n)}$ is given explicitly by Eq. (D6), while the solutions for the master integrals $\dot{\mathcal{X}}_{(n)}$, $\dot{\mathcal{Y}}_{(n)}$, $\dot{\mathcal{Z}}_{(n)}$ can be given in closed form by the recurrence relations in Eqs. (D7) - (D9) in Appendix D. The explicit form of $\dot{\mathcal{X}}_{(n)}$, $\dot{\mathcal{Y}}_{(n)}$, $\dot{\mathcal{Z}}_{(n)}$ for the first few orders in n are given by Eqs. (D10) - (D18) in Appendix D.

Two comments are in order: First of all, one may show, that $c\boldsymbol{\sigma} + \Delta\dot{\mathbf{x}}_{1\text{PN}}^{\text{M}} + \Delta\dot{\mathbf{x}}_{2\text{PN}}^{\text{M}\times\text{M}}$ agrees with Eq. (3.2.37) in [19] as well as with Eq. (44) in our article [79]; see also endnote [97]. And second, it has been checked numerically by the computer algebra system *Maple* [80] that the solutions in (80) - (85) satisfy the null condition (20) along the entire light trajectory, which represents an independent strict confirmation about the correctness of the time-independent coefficients given in Appendix E.

D. The second integration of geodesic equation in 2PN approximation

The second integration of geodesic equation (73) yields the trajectory of the light signal in 2PN approximation,

$$\begin{aligned}\mathbf{x}(t) &= \int_{-\infty}^t d\tau \frac{\dot{\mathbf{x}}(\tau)}{c} \\ &= \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\mathbf{x}_{1\text{PN}}(t, t_0) + \Delta\mathbf{x}_{2\text{PN}}(t, t_0) + \mathcal{O}(c^{-6}).\end{aligned}\quad (86)$$

By inserting (80) with (81) and (82) into the r.h.s. of (86) one gets

$$\Delta\mathbf{x}_{1\text{PN}}(t, t_0) = \Delta\mathbf{x}_{1\text{PN}}^{\text{M}}(t, t_0) + \Delta\mathbf{x}_{1\text{PN}}^{\text{Q}}(t, t_0), \quad (87)$$

$$\Delta\mathbf{x}_{2\text{PN}}(t, t_0) = \Delta\mathbf{x}_{2\text{PN}}^{\text{M}\times\text{M}}(t, t_0) + \Delta\mathbf{x}_{2\text{PN}}^{\text{M}\times\text{Q}}(t, t_0) + \Delta\mathbf{x}_{2\text{PN}}^{\text{Q}\times\text{Q}}(t, t_0). \quad (88)$$

The 1PN terms $\Delta \mathbf{x}_{1\text{PN}}^{\text{M}}$ and $\Delta \mathbf{x}_{1\text{PN}}^{\text{Q}}$ have been determined by Eqs. (66) and (67), respectively, and will not be re-presented here again.

The second integration of the 2PN monopole-monopole term in (75) is given by

$$\Delta x_{2\text{PN}}^{i\text{M}\times\text{M}}(t, t_0) = \frac{G^2 M^2}{c^4} \left[\sum_{n=3}^6 \mathcal{E}_{(n)}^i \mathcal{W}_{(n)}(t, t_0) + \sum_{n=2}^6 \mathcal{F}_{(n)}^i \mathcal{X}_{(n)}(t, t_0) + \mathcal{G}_{(5)}^i \mathcal{Y}_{(5)}(t, t_0) + \sum_{n=3,5} \mathcal{H}_{(n)}^i \mathcal{Z}_{(n)}(t, t_0) \right]. \quad (89)$$

The second integration of the 2PN monopole-quadrupole term in (78) is given by

$$\Delta x_{2\text{PN}}^{i\text{M}\times\text{Q}}(t, t_0) = \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \left[\sum_{n=3}^{10} \mathcal{K}_{(n)}^{iab} \mathcal{W}_{(n)}(t, t_0) + \sum_{n=2}^{10} \mathcal{L}_{(n)}^{iab} \mathcal{X}_{(n)}(t, t_0) + \sum_{n=7}^9 \mathcal{M}_{(n)}^{iab} \mathcal{Y}_{(n)}(t, t_0) + \sum_{n=5}^9 \mathcal{N}_{(n)}^{iab} \mathcal{Z}_{(n)}(t, t_0) \right]. \quad (90)$$

The second integration of the 2PN quadrupole-quadrupole term in (79) is given by

$$\Delta x_{2\text{PN}}^{i\text{Q}\times\text{Q}}(t, t_0) = \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \left[\sum_{n=5}^{14} \mathcal{P}_{(n)}^{iabcd} \mathcal{W}_{(n)}(t, t_0) + \sum_{n=4}^{14} \mathcal{Q}_{(n)}^{iabcd} \mathcal{X}_{(n)}(t, t_0) \right]. \quad (91)$$

The master integrals $\mathcal{W}_{(n)}$, $\mathcal{X}_{(n)}$, $\mathcal{Y}_{(n)}$, $\mathcal{Z}_{(n)}$ are defined by Eqs. (D20) - (D23). The solutions for the master integrals $\mathcal{W}_{(n)}$, $\mathcal{X}_{(n)}$, $\mathcal{Y}_{(n)}$, $\mathcal{Z}_{(n)}$ can be given in closed form by the recurrence relations in Eqs. (D25) - (D28) in Appendix D. The explicit form of $\mathcal{W}_{(n)}$, $\mathcal{X}_{(n)}$, $\mathcal{Y}_{(n)}$, $\mathcal{Z}_{(n)}$ for the first few orders in n are given by Eqs. (D29) - (D42) in Appendix D.

One comment is in order: One may show, that $\mathbf{x}_N + \Delta \mathbf{x}_{1\text{PN}}^{\text{M}} + \Delta \mathbf{x}_{2\text{PN}}^{\text{M}\times\text{M}}$ agrees with Eq. (3.2.38) in [19] as well as with Eq. (45) in our article [79]. After some algebraic manipulations, mainly using relation (34), one may also demonstrate that (89) agrees with Eq. (J11) in [37].

The presented solutions of the first integration (80) and second integration (86) of geodesic equation are strictly valid for any astrometric configuration between light source, body and observer, that means for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_0 > 0$ as well as for $\boldsymbol{\sigma} \cdot \mathbf{x}_1 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_1 > 0$.

VII. THE TOTAL LIGHT DEFLECTION

The solution of the first integration of the initial value problem allows to determine the total light deflection. For that, one has to consider the unit tangent vectors along the light trajectory at $t \rightarrow \pm\infty$,

$$\boldsymbol{\sigma} = \frac{\dot{\mathbf{x}}(t)}{c} \Big|_{t=-\infty}, \quad (92)$$

$$\boldsymbol{\nu} = \frac{\dot{\mathbf{x}}(t)}{c} \Big|_{t=+\infty}. \quad (93)$$

The total light deflection is defined as angle between these unit tangent vectors, $\delta(\boldsymbol{\sigma}, \boldsymbol{\nu})$, and the sine of total light deflection angle is given by

$$\sin \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}) = |\boldsymbol{\sigma} \times \boldsymbol{\nu}|. \quad (94)$$

From (80) - (82) with (62) - (63) and (83) - (85) one obtains for the unit tangent vector $\boldsymbol{\nu}$ in (93):

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \boldsymbol{\nu}_{1\text{PN}}^{\text{M}} + \boldsymbol{\nu}_{1\text{PN}}^{\text{Q}} + \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}} + \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}} + \boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}}, \quad (95)$$

$$\nu_{1\text{PN}}^{\text{M}} = \frac{GM}{c^2} \left[\mathcal{B}_{(3)}^i \dot{\chi}_{(3)}^\infty \right], \quad (96)$$

$$\nu_{1\text{PN}}^{\text{Q}} = \frac{G\hat{M}_{ab}}{c^2} \left[\sum_{n=5,7} \mathcal{D}_{(n)}^{i\ ab} \dot{\chi}_{(n)}^\infty \right], \quad (97)$$

$$\nu_{2\text{PN}}^{\text{M}\times\text{M}} = \frac{G^2 M^2}{c^4} \left[\sum_{n=2}^6 \mathcal{F}_{(n)}^i \dot{\chi}_{(n)}^\infty + \mathcal{G}_{(5)}^i \dot{\chi}_{(5)}^\infty \right], \quad (98)$$

$$\nu_{2\text{PN}}^{\text{M}\times\text{Q}} = \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \left[\sum_{n=2}^{10} \mathcal{L}_{(n)}^{i\ ab} \dot{\chi}_{(n)}^\infty + \sum_{n=7}^9 \mathcal{M}_{(n)}^{i\ ab} \dot{\chi}_{(n)}^\infty \right], \quad (99)$$

$$\nu_{2\text{PN}}^{\text{Q}\times\text{Q}} = \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \left[\sum_{n=4}^{14} \mathcal{Q}_{(n)}^{i\ abcd} \dot{\chi}_{(n)}^\infty \right], \quad (100)$$

where the abbreviations

$$\dot{\chi}_{(n)}^\infty = \dot{\chi}_{(n)}(t) \Big|_{t=+\infty} \quad \text{and} \quad \dot{\chi}_{(n)}^\infty = \dot{\chi}_{(n)}(t) \Big|_{t=+\infty} \quad (101)$$

have been introduced. It has been checked analytically that $\boldsymbol{\nu}$ in (95) is in fact a unit vector,

$$\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1 + \mathcal{O}(c^{-6}). \quad (102)$$

The result (102) demonstrates that the null condition (20) is satisfied along the light ray at future infinity up to terms of the order $\mathcal{O}(c^{-6})$ (note that the metric perturbations $h_{\alpha\beta}^{(n)}$ vanish at infinite distances from the body according to Eq. (9)). Furthermore, the proof of (102) implies an independent check about the correctness of those time-independent coefficients which arise in (95).

In (96) - (100) it has been taken into account that in the limit $t \rightarrow +\infty$ all terms with $\dot{\chi}_{(n)}$ vanish, while all terms with $\dot{\chi}_{(n)}$ cancel each other. The limits in (101) are given by Eqs. (D19) in Appendix D.

From (95) one obtains for the three-vector of total light deflection

$$\boldsymbol{\sigma} \times \boldsymbol{\nu} = \boldsymbol{\sigma} \times \boldsymbol{\nu}_{1\text{PN}}^{\text{M}} + \boldsymbol{\sigma} \times \boldsymbol{\nu}_{1\text{PN}}^{\text{Q}} + \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}} + \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}} + \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}}, \quad (103)$$

where the spatial components of the individual terms of the three-vector $\boldsymbol{\nu}$ are given by Eqs. (96) - (100). The total light deflection (103) is a coordinate-independent quantity and irrespective that it is calculated in a concrete coordinate system it is, in principle, an observable. The absolute value of the total light deflection (103) can be estimated by

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}| \leq |\boldsymbol{\sigma} \times \boldsymbol{\nu}_{1\text{PN}}^{\text{M}}| + |\boldsymbol{\sigma} \times \boldsymbol{\nu}_{1\text{PN}}^{\text{Q}}| + |\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}}| + |\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}}| + |\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}}|. \quad (104)$$

In order to determine the upper limit of each individual term in (104) the meaningful assumption is adopted, that to a good approximation the giant planets can be considered as axially symmetric bodies, where the STF quadrupole tensor (18) takes the following form [18, 81]:

$$\hat{M}_{ab} = M J_2 P^2 \frac{1}{3} \mathcal{R} \text{diag}(+1, +1, -2) \mathcal{R}^T, \quad (105)$$

where $\mathcal{R}$ is the rotational matrix giving the orientation of the unit-vector $\mathbf{e}_3$ of the symmetry axis of the massive body in the coordinate system, M is the mass of the body, J_2 is the actual second zonal harmonic coefficient, and P is the equatorial radius of the body. The parameters in (105) are given in Table I for the giant planets of the Solar System.

Table I: The numerical parameters: Schwarzschild radius $m = GM/c^2$ (in meter), equatorial radius P (in meter), and the actual second zonal harmonic coefficient J_2 of giant planets of the Solar System [82]. For later purposes in Section VIII the absolute value of the maximal distance $x_1^{\max}$ between the giant planet and a Gaia-like observer located at Lagrange point L_2 is given in astronomical units (au).

Object	m	P	J_2	$x_1^{\max}$
Jupiter	1.410	71.49×10^6	$14.697 \cdot 10^{-3}$	6
Saturn	0.422	60.27×10^6	$16.331 \cdot 10^{-3}$	11
Uranus	0.064	25.56×10^6	$3.516 \cdot 10^{-3}$	21
Neptune	0.076	24.76×10^6	$3.538 \cdot 10^{-3}$	31

By inserting (105) into (96) - (100) one obtains (details are given in Appendices F - I):

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{1\text{PN}}^{\text{M}}| \leq 4 \frac{GM}{c^2} \frac{1}{d_\sigma}, \quad (106)$$

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{1\text{PN}}^{\text{Q}}| \leq 4 \frac{GM}{c^2} |J_2| \frac{P^2}{(d_\sigma)^3}, \quad (107)$$

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M} \times \text{M}}| = \frac{G^2 M^2}{c^4} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} - 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|, \quad (108)$$

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M} \times \text{Q}}| \leq 4 \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{(d_\sigma)^2} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} + 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|, \quad (109)$$

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q} \times \text{Q}}| \leq 2 \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^4}{(d_\sigma)^4} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} + 12 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|. \quad (110)$$

These upper limits are strictly valid for any astrometric configuration between source, body and observer, that means for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_0 > 0$ as well as for $\boldsymbol{\sigma} \cdot \mathbf{x}_1 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_1 > 0$. The upper limits (106) - (107) take their maximal value in case of a light ray grazing at the surface of the massive body ($d_\sigma = P_A$). These maximal numerical values are presented in Table II for the giant planets of the Solar System.

Table II: The upper limits of the individual terms in (106) and (107) of the absolute value of total light deflection (104) for grazing light rays ($d_\sigma = P$) at the giant planets of the Solar System. All values are given in micro-arcseconds.

Object	$ \boldsymbol{\sigma} \times \boldsymbol{\nu}_{1\text{PN}}^{\text{M}} $	$ \boldsymbol{\sigma} \times \boldsymbol{\nu}_{1\text{PN}}^{\text{Q}} $
Jupiter	16272.7	239.2
Saturn	5776.9	94.3
Uranus	2065.9	7.3
Neptune	2532.5	9.0

If the massive body is, so to speak, located between the light source and the observer, then the following condition is valid,

$$\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0, \quad (111)$$

a condition that applies with certainty for any stellar light source. Then, the upper limits of the 2PN terms (108) - (110) simplify as follows

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M} \times \text{M}}| \leq \frac{15}{4} \pi \frac{G^2 M^2}{c^4} \frac{1}{(d_\sigma)^2}, \quad (112)$$

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M} \times \text{Q}}| \leq 15 \pi \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{(d_\sigma)^4}, \quad (113)$$

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q} \times \text{Q}}| \leq \frac{15}{2} \pi \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^4}{(d_\sigma)^6}. \quad (114)$$

The upper limits (112) - (114) take their maximal value in case of a light ray grazing at the surface of the massive body ($d_\sigma = P_A$). These maximal numerical values for such configurations are presented in Table III for the giant planets of the Solar System.

Table III: The upper limits of the individual terms in (112) - (114) of the absolute value of total light deflection (104) for grazing light rays ($d_\sigma = P$) at the giant planets of the Solar System. All values are given in nano-arcseconds.

Object	$ \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2PN}^{M \times M} $	$ \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2PN}^{M \times Q} $	$ \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2PN}^{Q \times Q} $
Jupiter	9.4×10^{-1}	5.5×10^{-2}	4.1×10^{-4}
Saturn	1.2×10^{-1}	7.8×10^{-3}	6.4×10^{-5}
Uranus	1.5×10^{-2}	2.1×10^{-4}	3.7×10^{-7}
Neptune	2.3×10^{-2}	3.3×10^{-4}	5.8×10^{-7}

In view of the small numerical values in Table III one might be inclined to believe that the light deflection caused by the quadrupole structure of any massive Solar System body is irrelevant in second post-Newtonian approximation for future astrometry missions on the sub-micro-arcsecond level and even on the nas-level of accuracy. This is, however, not true, because of the occurrence of so-called *enhanced 2PN terms*, which will be considered in the following Section.

VIII. ENHANCED TERMS

In the previous Section it has been shown that in case of condition (111) the 2PN terms in the total light deflection in Eqs. (108) - (110) can be estimated as $\text{const.} \times \frac{m^2}{d_\sigma^2}$. Therefore, they are so tiny that they can be neglected even for high-precision astrometry on the nano-arcsecond level; their numerical values are given in Table III for grazing light rays ($d_\sigma = P$). However, as worked out in several investigations [39, 40, 79] there are so-called *enhanced terms* of second post-Newtonian order in the theory of light propagation, whose upper limits are given by $\text{const.} \times \frac{m^2}{d_\sigma^2} \frac{x_1}{d_\sigma}$, where x_1 is the distance between massive body and observer and d_σ is the impact vector of the light ray. Hence $\frac{x_1}{d_\sigma} \gg 1$ for grazing rays where the impact vector equals the radius of the body. It is emphasized that these enhanced terms occur in case of condition (111), which is, for instance, always valid for light sources far away from the Solar System, like stellar light sources.

The occurrence of *enhanced terms* becomes apparent when the boundary value problem of light propagation is considered instead the initial value problem. Let $(ct_0, \mathbf{x}_0)$ be the four-coordinates of the light source at the moment of emission of the light signal, and $(ct_1, \mathbf{x}_1)$ be the four-coordinates of the observer at the moment of reception of the light signal. Then, the boundary value problem of light propagation is determined as solution of geodesic equation (19) with two boundary conditions

$$\mathbf{x}_0 = \mathbf{x}(t) \Big|_{t=t_0}, \quad (115)$$

$$\mathbf{x}_1 = \mathbf{x}(t) \Big|_{t=t_1}. \quad (116)$$

In reality an observer will not be located at infinity but at some finite distance from the massive Solar System body. Therefore, for relativistic modeling of real astrometric observations it is not sufficient to determine the initial value problem, defined by Eqs. (23) and (24), but the solution of the boundary value problem, defined by Eqs. (115) and (116), is required. This fact implies that for real astrometric measurements not the unit tangent-vector $\boldsymbol{\nu}$ at plus infinity is relevant, but the unit tangent-vector $\mathbf{n}$ at the time of reception, t_1 , that means at the exact position $\mathbf{x}_1$ of the observer (cf. Fig. 1). This unit tangent-vector is defined by

$$\mathbf{n} = \frac{\dot{\mathbf{x}}(t_1)}{|\dot{\mathbf{x}}(t_1)|}, \quad (117)$$

and the sine of the light deflection angle $\delta(\boldsymbol{\sigma}, \mathbf{n})$ at the observer's position is given by

$$\sin \delta(\boldsymbol{\sigma}, \mathbf{n}) = |\boldsymbol{\sigma} \times \mathbf{n}|. \quad (118)$$

The unit tangent-vector (117) is of fundamental importance for astrometry and implies that one has to introduce the impact vector $\hat{\mathbf{d}}_\sigma$ defined below by Eq. (130). Therefore, this impact vector becomes indispensable for modeling of real astrometric measurements and we will see that $\hat{\mathbf{d}}_\sigma$ is the reason for the occurrence of *enhanced terms* in the 2PN light deflection.

Now we consider the unit vector in (117). Inserting (80) into (117) yields for this vector in the 2PN approximation

$$\mathbf{n} = \boldsymbol{\sigma} + \mathbf{n}_{1\text{PN}}(\mathbf{x}_\text{N}) + \mathbf{n}_{2\text{PN}}(\mathbf{x}_\text{N}) + \mathcal{O}(c^{-6}), \quad (119)$$

where $\mathbf{n}_{1\text{PN}} = \mathcal{O}(c^{-2})$ and $\mathbf{n}_{2\text{PN}} = \mathcal{O}(c^{-4})$, which read

$$\mathbf{n}_{1\text{PN}} = \frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}}{c} - \frac{(\boldsymbol{\sigma} \cdot \Delta\dot{\mathbf{x}}_{1\text{PN}})}{c} \boldsymbol{\sigma}, \quad (120)$$

$$\mathbf{n}_{2\text{PN}} = \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}}{c} - \frac{(\boldsymbol{\sigma} \cdot \Delta\dot{\mathbf{x}}_{2\text{PN}})}{c} \boldsymbol{\sigma} - \frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}}{c} \left(\frac{\boldsymbol{\sigma} \cdot \Delta\dot{\mathbf{x}}_{1\text{PN}}}{c} \right) - \frac{1}{2} \left(\frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}}{c} \cdot \frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}}{c} \right) \boldsymbol{\sigma} + \frac{3}{2} \left(\frac{\boldsymbol{\sigma} \cdot \Delta\dot{\mathbf{x}}_{1\text{PN}}}{c} \right)^2 \boldsymbol{\sigma}, \quad (121)$$

where $\Delta\dot{\mathbf{x}}_{1\text{PN}}$ is given by Eqs. (61) - (63), while $\Delta\dot{\mathbf{x}}_{2\text{PN}}$ is given by Eqs. (82) - (85).

The 1PN terms $\mathbf{n}_{1\text{PN}}$ and the 2PN terms $\mathbf{n}_{2\text{PN}}$ in (119) depend on the spatial coordinates of the light source $\mathbf{x}_0$ as well as on the spatial coordinates of the unperturbed light signal at the moment of reception: $\mathbf{x}_\text{N}(t_1)$. However, for the boundary value problem the solution of the unit tangent-vector $\mathbf{n}$ must be given in terms of the position of the observer $\mathbf{x}_1$. But only in the 2PN terms $\mathbf{n}_{2\text{PN}}$ in (119) the argument $\mathbf{x}_\text{N}(t_1)$ can be replaced by the spatial position of the observer $\mathbf{x}_1$ because of

$$\mathbf{x}_1 = \mathbf{x}_\text{N}(t_1) + \mathcal{O}(c^{-2}), \quad (122)$$

hence such a replacement in $\mathbf{n}_{2\text{PN}}$ would cause an error of the order $\mathcal{O}(c^{-6})$ in Eq. (119) which is in line with the 2PN approximation. However, the argument $\mathbf{x}_\text{N}(t_1)$ of the 1PN terms $\mathbf{n}_{1\text{PN}}$ in (119) can not be replaced by $\mathbf{x}_1$, because such a replacement would cause an error of the order $\mathcal{O}(c^{-4})$ in (119) which would spoil the 2PN approximation. Therefore, one has to rewrite $\mathbf{n}_{1\text{PN}}$ in Eq. (119) in such a form that its argument is not the spatial coordinate of the unperturbed light signal, $\mathbf{x}_\text{N}(t_1)$, but given by the spatial coordinate of the light signal in 1PN approximation (64), that means $\mathbf{x}_{1\text{PN}}(t_1) = \mathbf{x}_\text{N}(t_1) + \Delta\mathbf{x}_{1\text{PN}}(t_1)$, where $\Delta\mathbf{x}_{1\text{PN}}$ is given by Eqs. (65) - (67). Then Eq. (119) reads

$$\mathbf{n} = \boldsymbol{\sigma} + \mathbf{n}_{1\text{PN}}(\mathbf{x}_{1\text{PN}}) + \mathbf{n}_{2\text{PN}}(\mathbf{x}_\text{N}) + \Delta\mathbf{n}_{2\text{PN}}(\mathbf{x}_\text{N}) + \mathcal{O}(c^{-6}), \quad (123)$$

where $\Delta\mathbf{n}_{2\text{PN}}$ is defined by

$$\Delta\mathbf{n}_{2\text{PN}}(\mathbf{x}_\text{N}) = \mathbf{n}_{1\text{PN}}(\mathbf{x}_\text{N}) - \mathbf{n}_{1\text{PN}}(\mathbf{x}_{1\text{PN}}) + \mathcal{O}(c^{-6}). \quad (124)$$

The expression for $\mathbf{n}_{1\text{PN}}(\mathbf{x}_\text{N})$ and $\mathbf{n}_{1\text{PN}}(\mathbf{x}_{1\text{PN}})$ are given by Eqs. (J1) and (J3) in Appendix J in its explicit form. Now, in Eq. (123) it is allowed to replace the argument in $\mathbf{n}_{1\text{PN}}(\mathbf{x}_{1\text{PN}})$ by the exact spatial position of the observer $\mathbf{x}_1$, because of

$$\mathbf{x}_1 = \mathbf{x}_{1\text{PN}}(t_1) + \mathcal{O}(c^{-4}), \quad (125)$$

which would cause an error of the order $\mathcal{O}(c^{-6})$ in (123) and would be in line with the 2PN approximation. To be consistent with the 2PN approximation, the additional term $\Delta\mathbf{n}_{2\text{PN}}$ in (124) has to be series expanded up to terms of the order $\mathcal{O}(c^{-6})$. The additional term $\Delta\mathbf{n}_{2\text{PN}}$ is of second post-Newtonian order $\mathcal{O}(c^{-4})$ and, in view of relation (122), in $\Delta\mathbf{n}_{2\text{PN}}$ one may replace $\mathbf{x}_\text{N}(t_1)$ by $\mathbf{x}_1$, that means the spatial position of the unperturbed light ray at the time of reception by the exact spatial position of the observer. Then, the unit tangent vector at the observers position follows from (123)

$$\mathbf{n} = \boldsymbol{\sigma} + \mathbf{n}_{1\text{PN}}(\mathbf{x}_1) + \mathbf{n}_{2\text{PN}}(\mathbf{x}_1) + \Delta\mathbf{n}_{2\text{PN}}(\mathbf{x}_1) + \mathcal{O}(c^{-6}). \quad (126)$$

Let us consider each individual term in (126). The expression $\mathbf{n}_{1\text{PN}}(\mathbf{x}_1)$ in (126) follows from (J3) when $\mathbf{x}_{1\text{PN}}(t_1)$ is

replaced by $\mathbf{x}_1$. Then, by inserting the multipole tensor (F3) into (J3) one obtains

$$\mathbf{n}_{1\text{PN}}(\mathbf{x}_1) = \mathbf{n}_{1\text{PN}}^{\text{M}}(\mathbf{x}_1) + \mathbf{n}_{1\text{PN}}^{\text{Q}}(\mathbf{x}_1), \quad (127)$$

$$\mathbf{n}_{1\text{PN}}^{\text{M}}(\mathbf{x}_1) = -2 \frac{GM}{c^2} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_1}{x_1}\right) \frac{\hat{\mathbf{d}}_\sigma}{(\hat{d}_\sigma)^2}, \quad (128)$$

$$\begin{aligned} \mathbf{n}_{1\text{PN}}^{\text{Q}}(\mathbf{x}_1) = & -2 \frac{GM}{c^2} J_2 \frac{P^2}{(\hat{d}_\sigma)^3} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_1}{x_1}\right) \left[\left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 - 4 \frac{(\hat{\mathbf{d}}_\sigma \cdot \mathbf{e}_3)^2}{(\hat{d}_\sigma)^2}\right) \frac{\hat{\mathbf{d}}_\sigma}{\hat{d}_\sigma} + 2 \frac{\hat{\mathbf{d}}_\sigma \cdot \mathbf{e}_3}{\hat{d}_\sigma} \mathbf{e}_3 \right] \\ & + \mathcal{O} \left(\frac{GM}{c^2} \frac{J_2}{\hat{d}_\sigma} \frac{P^2}{(x_1)^2} \right). \end{aligned} \quad (129)$$

In (129) terms proportional to $\boldsymbol{\sigma}$ have been omitted because they do not contribute to the light deflection angle in (118), while terms proportional to $\frac{P^2}{(x_1)^2} \ll 1$ have been neglected because they turn out to be much smaller than 1 nas. In the impact vector (J4) the spatial coordinate of the light signal in 1PN approximation $\mathbf{x}_{1\text{PN}}(t_1)$ can be replaced by the exact position of the observer $\mathbf{x}_1$ because of relation (125). Thus, the impact vector in (128) - (129) can be written in the form

$$\hat{\mathbf{d}}_\sigma = (\boldsymbol{\sigma} \times (\mathbf{x}_1 \times \boldsymbol{\sigma})) \quad \text{and} \quad \hat{d}_\sigma = |\hat{\mathbf{d}}_\sigma| \quad (130)$$

up to terms of the order $\mathcal{O}(c^{-4})$ which would cause an error of the order $\mathcal{O}(c^{-6})$ in (128) and (129) in line with the 2PN approximation; note that $\boldsymbol{\sigma} \cdot \hat{\mathbf{d}}_\sigma = 0$. The impact parameter $\hat{d}_\sigma$ is also larger than or equal to the equatorial radius P of the massive Solar System body

$$\hat{d}_\sigma \geq P. \quad (131)$$

The expression of the additional terms $\Delta \mathbf{n}_{2\text{PN}}(\mathbf{x}_1)$ in (126) is given by (see Eqs. (J10) - (J12) in Appendix J)

$$\Delta \mathbf{n}_{2\text{PN}} = \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{M}} + \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{Q}} + \Delta \mathbf{n}_{2\text{PN}}^{\text{Q} \times \text{Q}}, \quad (132)$$

$$\Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{M}}(\mathbf{x}_1) = 4 \frac{G^2 M^2}{c^4} \frac{1}{(\hat{d}_\sigma)^2} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_1}{x_1}\right) \frac{x_1 + \boldsymbol{\sigma} \cdot \mathbf{x}_1 - x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}{\hat{d}_\sigma} \frac{\hat{\mathbf{d}}_\sigma}{\hat{d}_\sigma}, \quad (133)$$

$$\begin{aligned} \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{Q}}(\mathbf{x}_1) = & 16 \frac{G^2 M^2}{c^4} J_2 \frac{P^2}{(\hat{d}_\sigma)^4} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_1}{x_1}\right) \frac{x_1 + \boldsymbol{\sigma} \cdot \mathbf{x}_1 - x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}{\hat{d}_\sigma} \\ & \times \left[\left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 - 3 \left(\frac{\hat{\mathbf{d}}_\sigma \cdot \mathbf{e}_3}{\hat{d}_\sigma}\right)^2\right) \frac{\hat{\mathbf{d}}_\sigma}{\hat{d}_\sigma} + \frac{\hat{\mathbf{d}}_\sigma \cdot \mathbf{e}_3}{\hat{d}_\sigma} \mathbf{e}_3 \right], \end{aligned} \quad (134)$$

$$\Delta \mathbf{n}_{2\text{PN}}^{\text{Q} \times \text{Q}}(\mathbf{x}_1) = 12 \frac{G^2 M^2}{c^4} (J_2)^2 \frac{P^4}{(\hat{d}_\sigma)^6} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_1}{x_1}\right) \frac{x_1 + \boldsymbol{\sigma} \cdot \mathbf{x}_1 - x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}{\hat{d}_\sigma} \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right)^2 \frac{\hat{\mathbf{d}}_\sigma}{\hat{d}_\sigma}, \quad (135)$$

where in (134) a term proportional to $\boldsymbol{\sigma}$ has been omitted because it does not contribute to the light deflection (118). In what follows it is shown that these terms are the so-called *enhanced 2PN terms*, because they are proportional to the large factor $\frac{x_1}{\hat{d}_\sigma}$, where x_1 is the distance between body and observer and $\hat{d}_\sigma$ equals the radius of the body for grazing light rays.

In Appendix J it is shown that these *enhanced 2PN terms* in Eqs. (133) - (135) are solely caused by the new impact vector (130) which is indispensable for modeling of real astrometric observations; cf. text below Eq. (J12) in Appendix J. These expressions are obtained from Eqs. (J10) - (J12) by replacing the spatial coordinate of the unperturbed light ray $\mathbf{x}_\text{N}(t_1)$ by the spatial coordinates of the observer $\mathbf{x}_1$. Such replacement causes an error of the order $\mathcal{O}(c^{-6})$, hence is in line with the 2PN approximation. We notice that in (133) - (135) the impact vector $\mathbf{d}_\sigma$ has been replaced by $\hat{\mathbf{d}}_\sigma$ which is in line with the 2PN approximation, because of

$$\hat{\mathbf{d}}_\sigma = \mathbf{d}_\sigma + \mathcal{O}(c^{-2}) \quad (136)$$

which results from (122). Finally, the term $\mathbf{n}_{2\text{PN}}(\mathbf{x}_1)$ in (126) is formally given by Eq. (121), when $\mathbf{x}_N(t_1)$ is replaced by $\mathbf{x}_1$.

By inspection of the unit vector in (126), one obtains for the upper limit of the light deflection (118)

$$|\boldsymbol{\sigma} \times \mathbf{n}| \leq |\boldsymbol{\sigma} \times \mathbf{n}_{1\text{PN}}(\mathbf{x}_1)| + |\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}(\mathbf{x}_1)| + |\boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}(\mathbf{x}_1)| + \mathcal{O}(c^{-6}). \quad (137)$$

Then, by using (127) - (129) and (132) - (135), as well as (121) with (81) - (82) one obtains for the upper limit (137) of the light deflection the following expression:

$$|\boldsymbol{\sigma} \times \mathbf{n}| \leq |\boldsymbol{\sigma} \times \mathbf{n}_{1\text{PN}}^{\text{M}}(\mathbf{x}_1)| + |\boldsymbol{\sigma} \times \mathbf{n}_{1\text{PN}}^{\text{Q}}(\mathbf{x}_1)| + |\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{M}}(\mathbf{x}_1)| + |\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{Q}}(\mathbf{x}_1)| + |\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{Q} \times \text{Q}}(\mathbf{x}_1)| \\ + |\boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{M}}(\mathbf{x}_1)| + |\boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{Q}}(\mathbf{x}_1)| + |\boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{Q} \times \text{Q}}(\mathbf{x}_1)| + \mathcal{O}(c^{-6}). \quad (138)$$

The 2PN terms in the second line of (138) are given by Eqs. (K3) - (K5) in Appendix K. The upper limits of each individual term in Eq. (138) read

$$|\boldsymbol{\sigma} \times \mathbf{n}_{1\text{PN}}^{\text{M}}| \leq 4 \frac{GM}{c^2} \frac{1}{\hat{d}_\sigma}, \quad (139)$$

$$|\boldsymbol{\sigma} \times \mathbf{n}_{1\text{PN}}^{\text{Q}}| \leq 4 \frac{GM}{c^2} |J_2| \frac{P^2}{(\hat{d}_\sigma)^3}, \quad (140)$$

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{M}}| \leq \frac{G^2 M^2}{c^4} \left| \frac{15}{4} \frac{\pi}{(\hat{d}_\sigma)^2} - 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(\hat{d}_\sigma)^3} \right|, \quad (141)$$

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{Q}}| \leq 4 \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{(\hat{d}_\sigma)^2} \left| \frac{15}{4} \frac{\pi}{(\hat{d}_\sigma)^2} + 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(\hat{d}_\sigma)^3} \right|, \quad (142)$$

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{Q} \times \text{Q}}| \leq 2 \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^4}{(\hat{d}_\sigma)^4} \left| \frac{15}{4} \frac{\pi}{(\hat{d}_\sigma)^2} + 12 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(\hat{d}_\sigma)^3} \right|, \quad (143)$$

$$|\boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{M}}| \leq 16 \frac{G^2 M^2}{c^4} \frac{1}{(\hat{d}_\sigma)^2} \frac{x_1}{\hat{d}_\sigma}, \quad (144)$$

$$|\boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{Q}}| \leq 64 \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{(\hat{d}_\sigma)^4} \frac{x_1}{\hat{d}_\sigma}, \quad (145)$$

$$|\boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{Q} \times \text{Q}}| \leq 48 \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^4}{(\hat{d}_\sigma)^6} \frac{x_1}{\hat{d}_\sigma}. \quad (146)$$

These upper limits are strictly valid for any astrometric configuration, that means for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_0 > 0$ as well as for $\boldsymbol{\sigma} \cdot \mathbf{x}_1 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_1 > 0$.

The estimation for the 1PN terms in (139) - (140) are not complicated; for instance (140) is shown by a very similar procedure as considered in Appendix F, while the upper limits for the 2PN terms (141) - (143) are considered in Appendix K. The upper limits for the *enhanced terms* in (144) - (146) follow straightforward from their expressions in (133) - (135) by using the relation (see endnote [98])

$$0 \leq x_1 + \boldsymbol{\sigma} \cdot \mathbf{x}_1 - x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq x_1 + \boldsymbol{\sigma} \cdot \mathbf{x}_1, \quad (147)$$

which is strictly valid for any astrometric configuration, that means for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_0 > 0$ as well as $\boldsymbol{\sigma} \cdot \mathbf{x}_1 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_1 > 0$. Furthermore, for the upper limit (145) relation (H10) has been applied.

Now let us consider the *enhanced terms* in (144) - (146). These terms are proportional to the large factor $\frac{x_1}{\hat{d}_\sigma}$, but this fact does not imply that these terms are divergent because their validity is restricted to the case of realistic observers located in the Solar System or at least having a distance of less than several hundred au from the massive Solar System body (see text below Eq. (J7)).

The *enhanced term* in (144) has already been obtained in case of 2PN light propagation in the monopole field some time ago [39, 40, 79]; e.g. relation (100) in [79] or relation(66) in [39] as well as Section 9 in [40]. The estimations in (145) and (146) are new results which demonstrate that *enhanced terms* do also arise in the case of 2PN light propagation in the quadrupole field. As mentioned, these *enhanced terms* originate because the light deflection in (118) is expressed in terms of the new impact vector (130).

There is an important difference between the 2PN terms in (141) - (143) and the *enhanced 2PN terms* in (144) - (146). Namely, if one considers again the most relevant astrometric configurations, where the body is, so to speak, located between the light source and the observer, that means where the condition (111) holds, then the 2PN terms in (141) - (143) simplify as follows

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{M}}| \leq \frac{15}{4} \pi \frac{G^2 M^2}{c^4} \frac{1}{(\hat{d}_\sigma)^2}, \quad (148)$$

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{Q}}| \leq 15 \pi \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{(\hat{d}_\sigma)^4}, \quad (149)$$

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2\text{PN}}^{\text{Q} \times \text{Q}}| \leq \frac{15}{2} \pi \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^4}{(\hat{d}_\sigma)^6}. \quad (150)$$

These terms are smaller than 1 nas for grazing light rays at the giant planets, hence they are negligible for astrometry on the sub-micro-arcsecond level and nano-arcsecond level of accuracy. In contrast, the upper limits for the 1PN terms (139) - (140) and the enhanced 2PN terms (144) - (146) remain unaffected and keep their validity under condition $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$.

For grazing light rays, $\hat{d}_\sigma = P_A$, the numerical magnitude of the upper limits (139) and (140) coincide with (106) and (107), respectively. Their numerical values were already given in Table II. Similarly, in order to quantify the numerical magnitude of the upper limits (144) - (146) we consider a light ray grazing at the surface of the massive body: $\hat{d}_\sigma = P_A$. The numerical values of these upper limits for such configurations are presented in Table IV for the giant planets of the Solar System.

Table IV: The upper limits of the *enhanced 2PN terms* of light deflection given by Eqs. (144) - (146) for grazing light rays ($\hat{d}_\sigma = P$) at the giant planets of the Solar System. For the distance x_1 a Gaia-like observer at Lagrange point L_2 is assumed and the maximal distances x_1^{max} between the giant planets and observer are used (see parameters given in Table I). All values are given in micro-arcseconds.

Object	$ \boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{M}} $	$ \boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{M} \times \text{Q}} $	$ \boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{Q} \times \text{Q}} $
Jupiter	16.11	0.95	0.010
Saturn	4.42	0.29	0.003
Uranus	2.58	0.04	0.001
Neptune	5.83	0.08	0.002

The upper limit of the *enhanced 2PN term* in (144) caused by the monopole structure is considerably larger (for numerical values see Table IV) than the upper limit of the 2PN monopole term of the total light deflection in (112) (for numerical values see Table III) and is already relevant on the μas -level of accuracy [39, 40, 79]. This fact has been taken into account in the refined model GREM [18, 79] which is in use for data reduction of the ESA astrometry mission Gaia [4–6].

Similarly, the upper limits of the *enhanced 2PN terms* in (145) and (146) caused by the quadrupole structure are considerably larger (for numerical values see Table IV) than the upper limits of the 2PN quadrupole terms in (113) and (114) of the total light deflection (for numerical values see Table III).

The *enhanced 2PN terms* of the quadrupole amount up to $0.95 \mu\text{as}$ and $0.29 \mu\text{as}$ for grazing rays at Jupiter and Saturn, hence they are still below the present state-of-the-art in astrometric measurements. For instance, the aimed accuracy of the ESA astrometry mission *Gaia* is about $5 \mu\text{as}$ in the final data release getting published after the end of mission expected around 2025 [9]. But in view of the impressive advancements of astrometric science during recent decades it is certain that future ground-based or space-based astrometric measurements aiming at the sub-micro-arcsecond level will be able to detect the quadrupole light deflection effect in 2PN approximation.

IX. SUMMARY AND OUTLOOK

The impressive progress in astrometric science makes it necessary to consider several subtle effects in the theory of light propagation. In particular, for high precision astrometry on the sub-micro-arcsecond level of accuracy 2PN effects need to be investigated. Thus far, however, only the impact of the monopole structure of massive bodies on 2PN light propagation has been considered in the literature. These 2PN effects of the monopole case have thoroughly been studied during recent years and in meanwhile a detailed understanding of these 2PN effects has been achieved.

On the other side, actually nothing is known about 2PN light deflection effects beyond the monopole structure of the massive bodies. For determining the geodesic equation in 2PN approximation one needs the metric of one body at rest in 2PN approximation. For that, the Multipole Post-Minkowskian (MPM) formalism has been used, which allows to determine the gravitational field of some massive body being of arbitrary shape, inner structure and rotational motion. According to the MPM approach, the gravitational field of some massive body is described by a multipole decomposition in terms of symmetric and trace-free mass-multipoles $\hat{M}_L$ (shape and inner structure of the body) and spin-multipoles $\hat{S}_L$ (rotational motions and inner circulations of the body).

In this investigation we have considered the problem of light propagation in 2PN approximation in the field of one body at rest with time-independent monopole and quadrupole structure, $\hat{M}_L = \{M, \hat{M}_{ab}\}$, while higher mass-multipoles as well as spin-multipoles of the body have been neglected. The primary results of this investigation are the following:

1. The coordinate velocity (first integration of geodesic equation) and the trajectory (second integration of geodesic equation) of a light signal has been determined in 2PN approximation, which propagates in the gravitational field of one body at rest, where the monopole structure and the quadrupole structure of the body has been taken into account. The first integration and the second integration can be reduced in each case to only four master integrals, Eqs. (D1) - (D4) and Eqs. (D20) - (D23). These integrals can be solved in closed form to any order by recurrence relations as given by Eqs. (D6) - (D9) and Eqs. (D25) - (D28).

2. The solutions for the first and second integration of geodesic equation are given by

$$\frac{\dot{\mathbf{x}}(t)}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\mathbf{x}}_{1\text{PN}}}{c} + \frac{\Delta\dot{\mathbf{x}}_{2\text{PN}}}{c} + \mathcal{O}(c^{-6}), \quad (151)$$

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PN}} + \Delta\mathbf{x}_{2\text{PN}} + \mathcal{O}(c^{-6}), \quad (152)$$

where the 1PN and 2PN corrections $\Delta\dot{\mathbf{x}}_{1\text{PN}}$ and $\Delta\dot{\mathbf{x}}_{2\text{PN}}$ are given by Eqs. (61) - (63) and Eqs. (82) - (85), respectively, while the 1PN and 2PN corrections $\Delta\mathbf{x}_{1\text{PN}}$ and $\Delta\mathbf{x}_{2\text{PN}}$ are given by Eqs. (65) - (67) and Eqs. (88) - (91), respectively. These solutions are valid for any astrometric configuration between source, body, and observer, that means for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_0 > 0$ as well as $\boldsymbol{\sigma} \cdot \mathbf{x}_1 \leq 1$ and $\boldsymbol{\sigma} \cdot \mathbf{x}_1 > 0$.

3. Using the given solution of the geodesic equation in 2PN approximation, the total light deflection has been determined in Eq. (103). The upper limits of the 1PN terms of the total light deflection are given by Eqs. (106) - (107) and their numerical values are given in Table II. The upper limits of the 2PN terms of the total light deflection are given by Eqs. (108) - (110). These upper limits are valid for any astrometric configuration between light source, massive body, and observer. For configurations with $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ (body is between source and observer) the upper limits of the 2PN terms of total light deflection are given by the inequalities (112) - (114) and turn out to be less than 1 nano-arcsecond as presented in Table III.

4. In view of the smallness of the 2PN terms in the total light deflection one might believe that the 2PN quadrupole corrections are negligible even for high-precision astrometry on the sub-micro-arcsecond level of accuracy. This is, however, not true. It is a well-known fact that there are *enhanced terms* for the case of 2PN light deflection in the gravitational field of one monopole at rest, which are proportional to $x_1/\hat{d}_\sigma$ and given by the inequality (144),

$$|\boldsymbol{\sigma} \times \Delta\mathbf{n}_{2\text{PN}}^{\text{M}\times\text{M}}| \leq 16 \frac{G^2 M^2}{c^4} \frac{1}{(\hat{d}_\sigma)^2} \frac{x_1}{\hat{d}_\sigma}, \quad (153)$$

where x_1 is the distance between massive body and observer and $\hat{d}_\sigma$ is the impact vector of the light ray. Hence $x_1/\hat{d}_\sigma \gg 1$ for grazing rays where the impact vector equals the radius of the body. These *enhanced 2PN terms* amount up to 16 micro-arcseconds for a grazing light ray at Jupiter and 4 micro-arcseconds for a grazing light ray at Saturn (cf. Table IV), besides that the total 2PN light deflection for one monopole at rest is less than 1 nano-arcsecond for grazing light rays at Jupiter [39, 40, 79].

Our considerations have revealed that there exist similar *enhanced 2PN terms* in the light deflection caused by the quadrupole structure of the body, which are also proportional to the large factor $x_1/\hat{d}_\sigma$. The upper limits of these

enhanced 2PN terms are given by the inequalities (145) - (146),

$$\left| \boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{M}\times\text{Q}} \right| \leq 64 \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{\left(\hat{d}_\sigma\right)^4} \frac{x_1}{\hat{d}_\sigma}, \quad (154)$$

$$\left| \boldsymbol{\sigma} \times \Delta \mathbf{n}_{2\text{PN}}^{\text{Q}\times\text{Q}} \right| \leq 48 \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^4}{\left(\hat{d}_\sigma\right)^6} \frac{x_1}{\hat{d}_\sigma}. \quad (155)$$

The enhanced 2PN quadrupole terms in (154) amount up to 0.95 micro-arcseconds for a grazing light ray at Jupiter and 0.29 micro-arcseconds for a grazing light ray at Saturn (cf. Table IV), which shows that they are relevant on the sub-micro-arcsecond level of astrometric accuracy. It has been emphasized that these *enhanced 2PN terms* in (144) - (146) are solely caused by the fact that the light deflection at observer's position in (118) is expressed in terms of the impact vector given by Eq. (130) which is indispensable for modeling of real astrometric measurements.

Finally, it is noticed that a set of several checks has been performed: (a) the calculations which lead to the time-independent coefficients (E41) - (E65) and (E67) - (E87) have been performed twice, namely analytically and by means of the computer algebra system *Maple* [80]. (b) it is a non-trivial fact that their prefactors are either integers or half-integral numbers. (c) it has been checked analytically that the tangent vector $\boldsymbol{\nu}$ in (95) is a unit vector. (d) it has been checked numerically by *Maple* that the null-condition (20) is satisfied along the entire light ray. (e) our results in case of 1PN and 2PN monopole as well as in case of 1PN quadrupole agree with known achievements in the literature.

Actually, one has to realize that at the sub-micro-arcsecond level of accuracy celestial light sources like stars and quasars, cannot be treated as pointlike but extended objects. Accordingly, one may separate the surface of celestial objects into sufficiently small surface elements such, that each individual surface element can then be treated as a pointlike light source. This treatment becomes more relevant the higher the aimed astrometric accuracies are.

Furthermore, in the limit of geometrical optics, light propagates along null-geodesics, governed by the geodesic equation (1) and the null-condition (2). In reality, however, light propagates as electromagnetic wave, and one has to investigate at which scale of accuracy the wave nature of light must be taken into account. Such an investigation has been performed in [87], where it was found that deviations from geometric optics are negligible even on the nano-arcsecond level, because the wavelength of electromagnetic waves in the optical band is much smaller than the curvature of space-time in the Solar System. In this respect we also refer to the recent investigations [88–90] where by using the eikonal approach the issue of image formation has been considered, which could be observed with telescopes located far out of the solar system.

The investigations presented are a basis for further considerations in the area of high-precision astrometry. In particular, in a subsequent investigation the boundary value problem of geodesic equation will be solved in 2PN approximation for the case of one body at rest with monopole and quadrupole structure, where also the Shapiro time delay will comprehensively be treated. Furthermore, the approach presented is also applicable to solve the problem of 2PN light propagation in the gravitational field of massive bodies with higher multipoles beyond the monopole and quadrupole structure of the bodies.

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Appendix A: Notation

Throughout the investigation the following notation is in use:

- G is the Newtonian constant of gravitation.
- c is the vacuum speed of light in Minkowskian space-time.

- M is the rest mass of the body defined by Eq. (17).
- $\hat{M}_{ab}$ is the quadrupole moment of the body defined by Eq. (18).
- $m = G M/c^2$ is the Schwarzschild radius of the body.
- P denotes the equatorial radius of the body.
- Lower case Latin indices $i, j, \dots$ take values 1, 2, 3.
- $\dot{f}$ denotes total time derivative of f .
- $f_{,i} = \partial f / \partial x^i$ denotes partial derivative of f with respect to x^i .
- $\delta_{ij} = \delta^{ij} = \text{diag}(+1, +1, +1)$ is Kronecker delta.
- $n! = n(n-1)(n-2) \cdots 2 \cdot 1$ is the factorial for positive integer ($0! = 1$).
- $n!! = n(n-2)(n-4) \cdots (2 \text{ or } 1)$ is the double factorial for positive integer ($0!! = 1$).
- $\varepsilon_{ijk} = \varepsilon^{ijk}$ with $\varepsilon_{123} = +1$ is the fully anti-symmetric Levi-Civita symbol.
- Triplet of three-vectors are in boldface: e.g. $\mathbf{a}, \mathbf{b}, \boldsymbol{\sigma}, \mathbf{x}$.
- Contravariant components of three-vectors: $a^i = (a^1, a^2, a^3)$.
- Absolute value of a three-vector: $a = |\mathbf{a}| = \sqrt{a^1 a^1 + a^2 a^2 + a^3 a^3}$.
- Scalar product of three-vectors: $\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a^i b^j$.
- Vector product of two three-vectors: $(\mathbf{a} \times \mathbf{b})^i = \varepsilon_{ijk} a^j b^k$.
- Angle between three-vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\delta(\mathbf{a}, \mathbf{b})$.
- Lower case Greek indices take values 0, 1, 2, 3.
- $f_{,\mu} = \partial f / \partial x^\mu$ denotes partial derivative of f with respect to x^μ .
- $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$ is the metric tensor of flat space-time.
- $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are the covariant and contravariant components of the metric tensor.
- Contravariant components of four-vectors: $a^\mu = (a^0, a^1, a^2, a^3)$.
- milli-arcsecond (mas): $1 \text{ mas} \simeq 4.85 \times 10^{-9} \text{ rad}$.
- micro-arcsecond (μas): $1 \mu\text{as} \simeq 4.85 \times 10^{-12} \text{ rad}$.
- nano-arcsecond (nas): $1 \text{ nas} \simeq 4.85 \times 10^{-15} \text{ rad}$.
- $\text{au} = 1.496 \times 10^{11} \text{ m}$ is the astronomical unit
- repeated indices are implicitly summed over (*Einstein's* sum convention).

Appendix B: The STF tensors $\hat{n}_L$

In this appendix we summarize some formulae of symmetric trace-free (STF) tensors which are relevant for our investigation. Further standard STF notation and some STF relations can be found in [61, 62, 83–86]; see also appendix B in [72]. The symmetric part of some tensor $\mathcal{J}^L$ with indices $L = i_1, \dots, i_l$ is given by

$$\mathcal{J}^{(L)} = \mathcal{J}^{(i_1 \dots i_l)} = \frac{1}{l!} \sum_{\sigma} \mathcal{J}^{i_{\sigma(1)} \dots i_{\sigma(l)}}, \quad (\text{B1})$$

where σ is running over all permutations of the indices $1, \dots, l$. If one subtracts all traces from (B1) then one gets the symmetric trace-free (STF) part of tensor $\mathcal{J}^L$. A general formula for the STF part of some tensor is provided by Eq. (2.2) in [61] or Eqs. (A 2a) - (A 2c) in [62]. Here, we will not repeat that formula, but just apply for a specific case of it, as given by Eq. (B6) below.

In particular, in our investigation the following Cartesian tensors are of primary importance,

$$x^L = x^{i_1} \dots x^{i_l} \quad \text{and} \quad n^L = \frac{x^{i_1}}{x} \dots \frac{x^{i_l}}{x} \quad (\text{B2})$$

where $L = i_1, \dots, i_l$ is a multi-index, x^i are the spatial coordinates of some arbitrary field point in space and $x = |\mathbf{x}|$. Note that $x_L = x^L$ and $n_L = n^L$. For instance,

$$n^{ab} = \frac{x^a x^b}{x^2}, \quad (\text{B3})$$

$$n^{abcd} = \frac{x^a x^b x^c x^d}{x^4}. \quad (\text{B4})$$

The symmetric trace-free (STF) part of these Cartesian tensors in (B2) are denoted by

$$\hat{x}^L = \text{STF}_{i_1 \dots i_l} x^{i_1} \dots x^{i_l} \quad \text{and} \quad \hat{n}^L = \text{STF}_{i_1 \dots i_l} \frac{x^{i_1}}{x} \dots \frac{x^{i_l}}{x} \quad (\text{B5})$$

where $\text{STF}_{i_1 \dots i_l}$ is the symmetrization and trace-free operation with respect to the spatial indices $i_1 \dots i_l$. Note that $\hat{x}_L = \hat{x}^L$ and $\hat{n}_L = \hat{n}^L$. A general formula to express $\hat{n}^L$ in terms of n^L is provided by (cf. Eq. (1.154) in [57])

$$\hat{n}^L = \sum_{k=0}^{[l/2]} (-1)^k \frac{(2l - 2k - 1)!!}{(2l - 1)!!} \frac{l!}{(l - 2k)!(2k)!!} \delta^{(i_1 i_2 \dots i_{2k-1} i_{2k}} n^{i_{2k+1} \dots i_l)} \quad (\text{B6})$$

where round brackets of the tensorial term indicate symmetrization according to Eq. (B1), double factorial is denoted by $m!!$ and $[l/2] = l/2$ for even l , while $[l/2] = (l - 1)/2$ for odd l . For instance,

$$\hat{n}^{ab} = n^{ab} - \frac{1}{3} \delta^{ab}, \quad (\text{B7})$$

$$\hat{n}^{abcd} = n^{abcd} - \frac{1}{7} (\delta^{ab} n^{cd} + \delta^{ac} n^{bd} + \delta^{ad} n^{bc} + \delta^{bc} n^{ad} + \delta^{bd} n^{ac} + \delta^{cd} n^{ab}) + \frac{1}{35} (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}), \quad (\text{B8})$$

$$\hat{n}^{abcdef} = n^{abcdef} - \frac{1}{11} (\delta^{ab} n^{cdef} + \text{sym.}) + \frac{1}{99} (\delta^{ab} \delta^{cd} n^{ef} + \text{sym.}) - \frac{1}{693} (\delta^{ab} \delta^{cd} \delta^{ef} + \text{sym.}). \quad (\text{B9})$$

These STF tensors appear in the 2PN metric coefficients in Eqs. (69) - (70). The expressions (B7) and (B8) were also given by Eqs. (1.153a) and (1.153c) in [57]; see also Eqs. (1.8.2) and (1.8.4) in [59]; the abbreviation *sym.* in (B9) means symmetrization with respect to the indices.

The geodesic equation in 1PN and 2PN approximation, Eqs. (44) and (71), contains spatial derivatives of the metric coefficients in (69) - (70). Therefore, one needs the spatial derivative of the STF tensor in (B6). The general formula is given by (cf. Eq. (A 24) in [62])

$$(\hat{n}^L)_{,k} = \frac{l+1}{x} n^k \hat{n}^L - \frac{2l+1}{x} \hat{n}^{kL}, \quad (\text{B10})$$

which can finally be expressed in terms of n^L by means of relation (B6).

As explained in the text below Eq. (71), that field point becomes the spatial coordinate of the light ray, which is expanded in terms of the unperturbed light ray; see Eqs. (E12) - (E14) in Appendix E. Therefore, one finally needs to consider the tensor n_L in (B2) and the STF tensor $\hat{n}_L$ in (B5) where the field point is replaced by the unperturbed light ray, that means

$$n_N^L = \frac{x_N^{i_1}}{x_N} \dots \frac{x_N^{i_l}}{x_N}, \quad (\text{B11})$$

and

$$\hat{n}_N^L = \text{STF}_{i_1 \dots i_l} \frac{x_N^{i_1}}{x_N} \dots \frac{x_N^{i_l}}{x_N}, \quad (\text{B12})$$

where x_N^i are the spatial coordinates of the unperturbed light ray as given by Eq. (31) and $x_N = |\mathbf{x}_N|$. From (B6) it is obvious that the general formula to express $\hat{n}_N^L$ in terms of n_N^L reads

$$\hat{n}_N^L = \sum_{k=0}^{[l/2]} (-1)^k \frac{(2l-2k-1)!!}{(2l-1)!!} \frac{l!}{(l-2k)!(2k)!!} \delta^{(i_1 i_2 \dots i_{2k-1} i_{2k} \dots i_{2k+1} \dots i_l)} n_N^{i_{2k+1} \dots i_l}. \quad (\text{B13})$$

Similarly, we need the spatial derivative of $\hat{n}^L$ in (B10), where (after derivation) the field point $\mathbf{x}$ is replaced by the unperturbed light ray $\mathbf{x}_N$, that means

$$(\hat{n}_N^L)_{,k} = \frac{l+1}{x} n_N^k \hat{n}_N^L - \frac{2l+1}{x} \hat{n}_N^{kL}. \quad (\text{B14})$$

By means of (B13) one may express such quantities in terms of n_N^L . In what follows, such terms are slightly imprecise called spatial derivative of $\hat{n}_N^L$, while these terms are actually spatial derivatives of $\hat{n}^L$ where after the differentiation the field point is replaced by the unperturbed light ray.

Appendix C: The auxiliary tensors T_N^{ijab} and $T_N^{iijabcd}$

These tensors T_N^{ijab} and $T_N^{iijabcd}$ are composed of STF tensors $\hat{n}_N^L$, nevertheless they themselves are neither symmetric in their indices nor trace-free. They are just abbreviations to simplify the notations. The auxiliary tensor T_N^{ijab} is defined by

$$T_N^{ijab} = \frac{15}{2} \hat{n}_N^{ijab} + \frac{32}{7} \delta^{ij} \hat{n}_N^{ab} - \frac{6}{7} \delta^{ai} \hat{n}_N^{jb} - \frac{6}{7} \delta^{aj} \hat{n}_N^{ib}, \quad (\text{C1})$$

where the STF tensors $\hat{n}_N^{ab}$ and $\hat{n}_N^{abcd}$ were given by Eq. (B13). The spatial derivative of T_N^{ijab} reads

$$(T_N^{ijab})_{,k} = \frac{15}{2} (\hat{n}_N^{ijab})_{,k} + \frac{32}{7} \delta^{ij} (\hat{n}_N^{ab})_{,k} - \frac{6}{7} \delta^{ai} (\hat{n}_N^{jb})_{,k} - \frac{6}{7} \delta^{aj} (\hat{n}_N^{ib})_{,k}, \quad (\text{C2})$$

where the terms on the r.h.s. are spatial derivatives of STF tensors and were given by Eq.(B14). The auxiliary tensor $T_N^{iijabcd}$ is defined by

$$\begin{aligned} T_N^{iijabcd} = & \frac{75}{4} \hat{n}_N^{iijabcd} - \frac{90}{11} \delta^{bd} \hat{n}_N^{ijac} + \frac{27}{11} \delta^{ij} \hat{n}_N^{abcd} - \frac{25}{84} \delta^{ac} \delta^{bd} \hat{n}_N^{ij} + \frac{83}{42} \delta^{bc} \delta^{ij} \hat{n}_N^{ad} + \frac{9}{11} \delta^{jb} \hat{n}_N^{acdi} \\ & + \frac{9}{11} \delta^{ib} \hat{n}_N^{acdj} - \frac{5}{42} \delta^{jc} \delta^{bd} \hat{n}_N^{ai} - \frac{5}{42} \delta^{ic} \delta^{bd} \hat{n}_N^{aj} + \frac{10}{21} \delta^{ci} \delta^{dj} \hat{n}_N^{ab} - \frac{23}{84} \delta^{bi} \delta^{jc} \hat{n}_N^{ad} - \frac{23}{84} \delta^{bj} \delta^{ic} \hat{n}_N^{ad}, \end{aligned} \quad (\text{C3})$$

where the STF tensors $\hat{n}_N^{ab}$, $\hat{n}_N^{abcd}$, and $\hat{n}_N^{abcdef}$ were given by Eq. (B13). The spatial derivative of $T_N^{iijabcd}$ reads

$$\begin{aligned} (T_N^{iijabcd})_{,k} = & \frac{75}{4} (\hat{n}_N^{iijabcd})_{,k} - \frac{90}{11} \delta^{bd} (\hat{n}_N^{ijac})_{,k} + \frac{27}{11} \delta^{ij} (\hat{n}_N^{abcd})_{,k} - \frac{25}{84} \delta^{ac} \delta^{bd} (\hat{n}_N^{ij})_{,k} \\ & + \frac{83}{42} \delta^{bc} \delta^{ij} (\hat{n}_N^{ad})_{,k} + \frac{9}{11} \delta^{jb} (\hat{n}_N^{acdi})_{,k} + \frac{9}{11} \delta^{ib} (\hat{n}_N^{acdj})_{,k} - \frac{5}{42} \delta^{jc} \delta^{bd} (\hat{n}_N^{ai})_{,k} \\ & - \frac{5}{42} \delta^{ic} \delta^{bd} (\hat{n}_N^{aj})_{,k} + \frac{10}{21} \delta^{ci} \delta^{dj} (\hat{n}_N^{ab})_{,k} - \frac{23}{84} \delta^{bi} \delta^{jc} (\hat{n}_N^{ad})_{,k} - \frac{23}{84} \delta^{bj} \delta^{ic} (\hat{n}_N^{ad})_{,k}, \end{aligned} \quad (\text{C4})$$

where the terms on the r.h.s. are spatial derivatives of STF tensors and were given by Eq. (B14).

Appendix D: Scalar master integrals

1. Scalar master integrals for the first integration

The first integration of geodesic equation in 1PN and 2PN approximation can be reduced to finally only four kind of scalar master integrals,

$$\dot{\mathcal{W}}_{(n)}(t) = \int_{-\infty}^t d\mathbf{c}t \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^n}, \quad (\text{D1})$$

$$\dot{\mathcal{X}}_{(n)}(t) = \int_{-\infty}^t d\mathbf{c}t \frac{1}{(x_N)^n}, \quad (\text{D2})$$

$$\dot{\mathcal{Y}}_{(n)}(t) = \int_{-\infty}^t d\mathbf{c}t \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^n} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}, \quad (\text{D3})$$

$$\dot{\mathcal{Z}}_{(n)}(t) = \int_{-\infty}^t d\mathbf{c}t \frac{1}{(x_N)^n} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}, \quad (\text{D4})$$

where $\mathbf{x}_N = \mathbf{x}_N(t)$ and $x_N = x_N(t)$ are functions of the integration variable t according to (31) and (32). Clearly, the limits

$$\lim_{t \rightarrow -\infty} \dot{\mathcal{W}}_{(n)}(t) = 0, \quad \lim_{t \rightarrow -\infty} \dot{\mathcal{X}}_{(n)}(t) = 0, \quad \lim_{t \rightarrow -\infty} \dot{\mathcal{Y}}_{(n)}(t) = 0, \quad \lim_{t \rightarrow -\infty} \dot{\mathcal{Z}}_{(n)}(t) = 0, \quad (\text{D5})$$

are in accordance with relation (26). The scalar master integrals for the first integration can be given in closed form,

$$\dot{\mathcal{W}}_{(n)}(t) = -\frac{1}{n-2} \frac{1}{(x_N)^{n-2}}, \quad (\text{D6})$$

$$\dot{\mathcal{X}}_{(n)}(t) = \frac{1}{n-2} \frac{1}{(d_\sigma)^2} \left((n-3) \dot{\mathcal{X}}_{(n-2)}(t) + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^{n-2}} \right), \quad (\text{D7})$$

$$\dot{\mathcal{Y}}_{(n)}(t) = -\frac{1}{n-2} \left(\dot{\mathcal{X}}_{(n-1)}(t) + \frac{1}{(x_N)^{n-2}} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} \right), \quad (\text{D8})$$

$$\dot{\mathcal{Z}}_{(n)}(t) = \frac{1}{n-2} \frac{1}{(d_\sigma)^2} \left((n-3) \dot{\mathcal{Z}}_{(n-2)}(t) - \frac{1}{n-3} \frac{1}{(x_N)^{n-3}} + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^{n-2}} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} \right), \quad (\text{D9})$$

where $n \geq 3$ in Eqs. (D6) and (D8) and $n \geq 4$ in Eqs. (D7) and (D9). While (D6) represents the explicit solution of the integrals (D1) to any order, the implicit recurrence relations (D7) - (D9) allow us to determine the explicit solutions of the integrals (D2) - (D4) to any order of the inverse powers in n as soon as the first orders of them are known, which have to be determined directly by using standard integrals. Furthermore, they are very useful for estimating their upper limits of impact on light deflection and they can be used to make some cross checks of the explicit solutions for the integrals. These integrals (D6) - (D9) are stable, that means they are convergent with respect to the limit $n \rightarrow \infty$. The explicit solutions for the first few orders of the inverse powers in n are given by

$$\dot{\mathcal{X}}_{(2)}(t) = \frac{1}{d_\sigma} \left(\arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} + \frac{\pi}{2} \right), \quad (\text{D10})$$

$$\dot{\mathcal{X}}_{(3)}(t) = \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} + 1 \right), \quad (\text{D11})$$

$$\dot{\mathcal{X}}_{(4)}(t) = \frac{1}{2} \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^2} + \frac{1}{d_\sigma} \left(\arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} + \frac{\pi}{2} \right) \right), \quad (\text{D12})$$

$$\dot{\chi}_{(5)}(t) = \frac{2}{3} \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} \frac{1}{(d_\sigma)^2} + \frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} + \frac{1}{(d_\sigma)^2} \right), \quad (\text{D13})$$

$$\dot{\chi}_{(6)}(t) = \frac{1}{4} \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^4} + \frac{3}{2} \frac{1}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^2} + \frac{3}{2} \frac{1}{(d_\sigma)^3} \left(\arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} + \frac{\pi}{2} \right) \right), \quad (\text{D14})$$

$$\dot{\chi}_{(7)}(t) = \frac{8}{15} \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} \frac{1}{(d_\sigma)^4} + \frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} \frac{1}{(d_\sigma)^2} + \frac{3}{8} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^5} + \frac{1}{(d_\sigma)^4} \right), \quad (\text{D15})$$

$$\dot{\chi}_{(5)}(t) = -\frac{1}{3} \frac{1}{(x_N)^3} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} - \frac{1}{6} \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^2} + \frac{1}{d_\sigma} \left(\arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} + \frac{\pi}{2} \right) \right), \quad (\text{D16})$$

$$\dot{\chi}_{(3)}(t) = \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} + 1 \right) \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} - \frac{1}{(d_\sigma)^2} \left(\ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{2x_N} \right), \quad (\text{D17})$$

$$\dot{\chi}_{(5)}(t) = \frac{2}{3} \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} \frac{1}{(d_\sigma)^2} + \frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} + \frac{1}{(d_\sigma)^2} \right) \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} - \frac{1}{6} \frac{1}{(d_\sigma)^2} \frac{1}{(x_N)^2} - \frac{2}{3} \frac{1}{(d_\sigma)^4} \left(\ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{2x_N} \right), \quad (\text{D18})$$

where the time arguments have been omitted in the terms on the r.h.s. in order to simplify the notation.

For the expression of the total light deflection one needs the limits in (101). Using $\lim_{t \rightarrow +\infty} \boldsymbol{\sigma} \cdot \mathbf{x}_N / x_N = 1$, which follows from (31) and (32), one obtains:

$$\begin{aligned} \dot{\chi}_{(2)}^\infty &= \frac{\pi}{d_\sigma}, \quad \dot{\chi}_{(3)}^\infty = \frac{2}{(d_\sigma)^2}, \quad \dot{\chi}_{(4)}^\infty = \frac{\pi}{2} \frac{1}{(d_\sigma)^3}, \quad \dot{\chi}_{(5)}^\infty = \frac{4}{3} \frac{1}{(d_\sigma)^4}, \quad \dot{\chi}_{(6)}^\infty = \frac{3}{8} \frac{\pi}{(d_\sigma)^5}, \quad \dot{\chi}_{(7)}^\infty = \frac{16}{15} \frac{1}{(d_\sigma)^6}, \\ \dot{\chi}_{(8)}^\infty &= \frac{5}{16} \frac{\pi}{(d_\sigma)^7}, \quad \dot{\chi}_{(9)}^\infty = \frac{32}{35} \frac{1}{(d_\sigma)^8}, \quad \dot{\chi}_{(10)}^\infty = \frac{35}{128} \frac{\pi}{(d_\sigma)^9}, \quad \dot{\chi}_{(11)}^\infty = \frac{256}{315} \frac{1}{(d_\sigma)^{10}}, \quad \dot{\chi}_{(12)}^\infty = \frac{63}{256} \frac{\pi}{(d_\sigma)^{11}}, \\ \dot{\chi}_{(13)}^\infty &= \frac{512}{693} \frac{1}{(d_\sigma)^{12}}, \quad \dot{\chi}_{(14)}^\infty = \frac{231}{1024} \frac{\pi}{(d_\sigma)^{13}}, \quad \dot{\chi}_{(5)}^\infty = -\frac{1}{6} \frac{\pi}{(d_\sigma)^3}, \quad \dot{\chi}_{(7)}^\infty = -\frac{3}{40} \frac{\pi}{(d_\sigma)^5}, \quad \dot{\chi}_{(9)}^\infty = -\frac{5}{112} \frac{\pi}{(d_\sigma)^7}. \end{aligned} \quad (\text{D19})$$

2. Scalar master integrals for the second integration

The second integration of geodesic equation in 1PN and 2PN approximation can be reduced to finally only four scalar master integrals,

$$\mathcal{W}_{(n)}(t, t_0) = \int_{t_0}^t dt \dot{\mathcal{W}}_{(n)}(t) = \mathcal{W}_{(n)}(t) - \mathcal{W}_{(n)}(t_0), \quad (\text{D20})$$

$$\mathcal{X}_{(n)}(t, t_0) = \int_{t_0}^t dt \dot{\mathcal{X}}_{(n)}(t) = \mathcal{X}_{(n)}(t) - \mathcal{X}_{(n)}(t_0), \quad (\text{D21})$$

$$\mathcal{Y}_{(n)}(t, t_0) = \int_{t_0}^t dt \dot{\mathcal{Y}}_{(n)}(t) = \mathcal{Y}_{(n)}(t) - \mathcal{Y}_{(n)}(t_0), \quad (\text{D22})$$

$$\mathcal{Z}_{(n)}(t, t_0) = \int_{t_0}^t dt \dot{\mathcal{Z}}_{(n)}(t) = \mathcal{Z}_{(n)}(t) - \mathcal{Z}_{(n)}(t_0). \quad (\text{D23})$$

Clearly, the limits

$$\lim_{t \rightarrow t_0} \mathcal{W}_{(n)}(t, t_0) = 0, \quad \lim_{t \rightarrow t_0} \mathcal{X}_{(n)}(t, t_0) = 0, \quad \lim_{t \rightarrow t_0} \mathcal{Y}_{(n)}(t, t_0) = 0, \quad \lim_{t \rightarrow t_0} \mathcal{Z}_{(n)}(t, t_0) = 0, \quad (\text{D24})$$

are in accordance with relation (29). The scalar master integrals for the second integration can be given in closed form,

$$\mathcal{W}_{(n)}(t) = -\frac{1}{n-2} \dot{\mathcal{X}}_{(n-2)}(t), \quad (\text{D25})$$

$$\mathcal{X}_{(n)}(t) = \frac{1}{n-2} \frac{1}{(d_\sigma)^2} \left(\dot{\mathcal{W}}_{(n-2)}(t) + (n-3) \mathcal{X}_{(n-2)}(t) \right), \quad (\text{D26})$$

$$\mathcal{Y}_{(n)}(t) = -\frac{1}{n-2} \left(\dot{\mathcal{Z}}_{(n-2)}(t) + \mathcal{X}_{(n-1)}(t) \right), \quad (\text{D27})$$

$$\mathcal{Z}_{(n)}(t) = \frac{1}{n-2} \frac{1}{(d_\sigma)^2} \left((n-3) \mathcal{Z}_{(n-2)}(t) - \frac{1}{n-3} \dot{\mathcal{X}}_{(n-3)}(t) + \dot{\mathcal{Y}}_{(n-2)}(t) \right), \quad (\text{D28})$$

where $n \geq 4$ in Eq. (D25) and $n \geq 5$ in Eqs. (D26) - (D28). The recurrence relations (D25) - (D28) allow us to determine the solutions of the integrals (D20) - (D23) to any order of the power n as soon as the very few orders of them are known, which have to be determined directly by using standard integrals. Furthermore, they are very useful for estimating their upper limits of impact on time delay and light deflection and they can be used to cross check some of the explicit solutions for the integrals. These integrals (D25) - (D28) are stable, that means they are convergent in the limit $n \rightarrow \infty$. The explicit solutions for the first few orders of the power in n are given by (constants which cancel each other due to (D20) - (D23) have been omitted)

$$\mathcal{W}_{(3)}(t) = \ln(x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N), \quad (\text{D29})$$

$$\mathcal{W}_{(4)}(t) = -\frac{1}{2} \frac{1}{d_\sigma} \arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma}, \quad (\text{D30})$$

$$\mathcal{W}_{(5)}(t) = -\frac{1}{3} \frac{1}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N}, \quad (\text{D31})$$

$$\mathcal{W}_{(6)}(t) = -\frac{1}{8} \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^2} + \frac{1}{d_\sigma} \arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} \right), \quad (\text{D32})$$

$$\mathcal{W}_{(7)}(t) = -\frac{2}{15} \frac{1}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} \frac{1}{(d_\sigma)^2} + \frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} \right), \quad (\text{D33})$$

$$\mathcal{X}_{(2)}(t) = +\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} \left(\arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} + \frac{\pi}{2} \right) - \ln x_N, \quad (\text{D34})$$

$$\mathcal{X}_{(3)}(t) = +\frac{1}{(d_\sigma)^2} (x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N), \quad (\text{D35})$$

$$\mathcal{X}_{(4)}(t) = +\frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(d_\sigma)^3} \left(\arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} + \frac{\pi}{2} \right), \quad (\text{D36})$$

$$\mathcal{X}_{(5)}(t) = +\frac{2}{3} \frac{1}{(d_\sigma)^2} \left(\frac{x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N}{(d_\sigma)^2} - \frac{1}{2} \frac{1}{x_N} \right), \quad (\text{D37})$$

$$\mathcal{X}_{(6)}(t) = -\frac{1}{8} \frac{1}{(d_\sigma)^2} \left(\frac{1}{(x_N)^2} - 3 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(d_\sigma)^3} \left(\arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} + \frac{\pi}{2} \right) \right), \quad (\text{D38})$$

$$\mathcal{X}_{(7)}(t) = +\frac{8}{15} \frac{1}{(d_\sigma)^2} \left(\frac{x_N}{(d_\sigma)^4} - \frac{1}{2} \frac{1}{x_N} \frac{1}{(d_\sigma)^2} - \frac{1}{8} \frac{1}{(x_N)^3} + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(d_\sigma)^4} \right), \quad (\text{D39})$$

$$\mathcal{Y}_{(5)}(t) = -\frac{1}{3} \frac{1}{(d_\sigma)^2} \left(\left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} + 1 \right) \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} + \frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} \left(\arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} + \frac{\pi}{2} \right) \right) + \frac{1}{3} \frac{1}{(d_\sigma)^2} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{2 x_N}, \quad (\text{D40})$$

$$\mathcal{Z}_{(3)}(t) = \frac{x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N}{(d_\sigma)^2} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} + \frac{1}{d_\sigma} \arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} - \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(d_\sigma)^2} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{2 x_N}, \quad (\text{D41})$$

$$\mathcal{Z}_{(5)}(t) = \frac{2}{3} \frac{1}{(d_\sigma)^2} \left(\frac{x_N}{(d_\sigma)^2} + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(d_\sigma)^2} - \frac{1}{2} \frac{1}{x_N} \right) \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} + \frac{1}{6} \frac{1}{(d_\sigma)^3} \arctan \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{d_\sigma} - \frac{2}{3} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(d_\sigma)^4} \ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{2 x_N}, \quad (\text{D42})$$

where the time arguments have been omitted in order to simplify the notation. Note that terms like $\ln(x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N)$ in (D29) and $\ln x_N$ in (D34) are well-defined in view of (D20) and (D21).

Appendix E: The terms of 2PN geodesic equation

In this appendix each term of the 2PN geodesic equation (72) is given, which can be written in the following form:

$$\frac{\ddot{x}^i}{c^2} = \frac{\ddot{x}_A^i}{c^2} + \frac{\ddot{x}_B^i}{c^2} + \frac{\ddot{x}_C^i}{c^2} + \frac{\ddot{x}_D^i}{c^2} + \frac{\ddot{x}_E^i}{c^2} + \frac{\ddot{x}_F^i}{c^2} + \frac{\ddot{x}_G^i}{c^2} + \frac{\ddot{x}_H^i}{c^2} + \frac{\ddot{x}_I^i}{c^2} + \frac{\ddot{x}_J^i}{c^2} + \mathcal{O}(c^{-6}), \quad (\text{E1})$$

where the individual terms are

$$\frac{\ddot{x}_A^i}{c^2} = h_{00,i}^{(2)}(\mathbf{x}_{1\text{PN}}), \quad (\text{E2})$$

$$\frac{\ddot{x}_B^i}{c^2} = -2 h_{00,j}^{(2)}(\mathbf{x}_{1\text{PN}}) \sigma^i \sigma^j, \quad (\text{E3})$$

$$\frac{\ddot{x}_C^i}{c^2} = h_{00,i}^{(2)}(\mathbf{x}_N) \frac{\sigma^j \Delta \dot{x}_{1\text{PN}}^j}{c}, \quad (\text{E4})$$

$$\frac{\ddot{x}_D^i}{c^2} = -2 h_{00,j}^{(2)}(\mathbf{x}_N) \frac{\Delta \dot{x}_{1\text{PN}}^j}{c} \sigma^i, \quad (\text{E5})$$

$$\frac{\ddot{x}_E^i}{c^2} = -2 h_{00,j}^{(2)}(\mathbf{x}_N) \frac{\Delta \dot{x}_{1\text{PN}}^i}{c} \sigma^j, \quad (\text{E6})$$

$$\frac{\ddot{x}_F^i}{c^2} = -h_{00}^{(2)}(\mathbf{x}_N) h_{00,i}^{(2)}(\mathbf{x}_N), \quad (\text{E7})$$

$$\frac{\ddot{x}_G^i}{c^2} = \frac{1}{2} h_{00,i}^{(4)}(\mathbf{x}_N), \quad (\text{E8})$$

$$\frac{\ddot{x}_H^i}{c^2} = -h_{00,j}^{(4)}(\mathbf{x}_N) \sigma^i \sigma^j, \quad (\text{E9})$$

$$\frac{\ddot{x}_I^i}{c^2} = -h_{ij,k}^{(4)}(\mathbf{x}_N) \sigma^j \sigma^k, \quad (\text{E10})$$

$$\frac{\ddot{x}_J^i}{c^2} = \frac{1}{2} h_{jk,i}^{(4)}(\mathbf{x}_N) \sigma^j \sigma^k. \quad (\text{E11})$$

The following subsections are devoted to each of these 10 individual terms. In what follows, one needs the spatial coordinates and the inverse powers of their absolute values in 1PN approximation, that is, up to terms of the order $\mathcal{O}(c^{-4})$:

$$\mathbf{x}_{1\text{PN}} = \mathbf{x}_N + \Delta\mathbf{x}_{1\text{PN}}, \quad (\text{E12})$$

$$x_{1\text{PN}} = x_N + \frac{\mathbf{x}_N \cdot \Delta\mathbf{x}_{1\text{PN}}}{x_N}, \quad (\text{E13})$$

$$\frac{1}{(x_{1\text{PN}})^n} = \frac{1}{(x_N)^n} - \frac{n}{(x_N)^n} \frac{\mathbf{x}_N \cdot \Delta\mathbf{x}_{1\text{PN}}}{(x_N)^2}, \quad (\text{E14})$$

where Eq. (E12) was given by Eq. (64) and expression $\Delta\mathbf{x}_{1\text{PN}}$ is given by Eqs. (65) - (67).

1. The term $\ddot{x}_A^i$

Determining the spatial derivatives of $h_{00}^{(2)}(\mathbf{x})$ in (41) and taking $\mathbf{x} = \mathbf{x}_{1\text{PN}} + \mathcal{O}(c^{-4})$ one gets

$$\frac{\ddot{x}_A^i}{c^2} = -2 \frac{GM}{c^2} \frac{x_{1\text{PN}}^i}{(x_{1\text{PN}})^3} + 6 \frac{G\hat{M}_{ai}}{c^2} \frac{x_{1\text{PN}}^a}{(x_{1\text{PN}})^5} - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_{1\text{PN}}^a x_{1\text{PN}}^b x_{1\text{PN}}^i}{(x_{1\text{PN}})^7}. \quad (\text{E15})$$

By means of (E12) and (E14) one obtains

$$\begin{aligned} \frac{\ddot{x}_A^i}{c^2} = & -2 \frac{GM}{c^2} \frac{x_N^i}{(x_N)^3} + 6 \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^5} - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^7} - 2 \frac{GM}{c^2} \frac{\Delta x_{1\text{PN}}^i}{(x_N)^3} + 6 \frac{GM}{c^2} \frac{x_N^i}{(x_N)^5} (\mathbf{x}_N \cdot \Delta\mathbf{x}_{1\text{PN}}) \\ & + 6 \frac{G\hat{M}_{ai}}{c^2} \frac{\Delta x_{1\text{PN}}^a}{(x_N)^5} - 30 \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^7} (\mathbf{x}_N \cdot \Delta\mathbf{x}_{1\text{PN}}) - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b}{(x_N)^7} \Delta x_{1\text{PN}}^i \\ & - 30 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^i}{(x_N)^7} \Delta x_{1\text{PN}}^b + 105 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^9} (\mathbf{x}_N \cdot \Delta\mathbf{x}_{1\text{PN}}), \end{aligned} \quad (\text{E16})$$

where the symmetry $\hat{M}_{ab} = \hat{M}_{ba}$ has been used in the next-to-last term. According to Eq. (65), the term $\Delta\mathbf{x}_{1\text{PN}}$ is separated into a monopole and quadrupole term, $\Delta\mathbf{x}_{1\text{PN}} = \Delta\mathbf{x}_{1\text{PN}}^M + \Delta\mathbf{x}_{1\text{PN}}^Q$, given by Eq. (66) and (67), respectively.

2. The term $\ddot{x}_B^i$

Determining the spatial derivatives of $h_{00}^{(2)}(\mathbf{x})$ in (41) and taking $\mathbf{x} = \mathbf{x}_{1\text{PN}}$ one gets

$$\frac{\ddot{x}_B^i}{c^2} = 4 \frac{GM}{c^2} \sigma^a \sigma^i \frac{x_{1\text{PN}}^a}{(x_{1\text{PN}})^3} - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \sigma^i \frac{x_{1\text{PN}}^a}{(x_{1\text{PN}})^5} + 30 \frac{G\hat{M}_{ab}}{c^2} \sigma^c \sigma^i \frac{x_{1\text{PN}}^a x_{1\text{PN}}^b x_{1\text{PN}}^c}{(x_{1\text{PN}})^7}. \quad (\text{E17})$$

By inserting the spatial coordinates and their absolute values in 1PN approximation (E12) - (E14) one obtains

$$\begin{aligned} \frac{\ddot{x}_B^i}{c^2} = & 4 \frac{GM}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \sigma^i \frac{x_N^a}{(x_N)^5} + 30 \frac{G\hat{M}_{ab}}{c^2} \sigma^i x_N^a x_N^b \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^7} + 4 \frac{GM}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{1\text{PN}}}{(x_N)^3} \\ & - 12 \frac{GM}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^5} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}) - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \sigma^i \frac{\Delta x_{1\text{PN}}^a}{(x_N)^5} + 60 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \sigma^i \frac{x_N^a}{(x_N)^7} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}), \\ & + 30 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{x_N^a x_N^b}{(x_N)^7} (\boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{1\text{PN}}) + 60 \frac{G\hat{M}_{ab}}{c^2} \sigma^i x_N^a \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^7} \Delta x_{1\text{PN}}^b - 210 \frac{G\hat{M}_{ab}}{c^2} \sigma^i x_N^a x_N^b \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^9} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}), \end{aligned} \quad (\text{E18})$$

where the symmetry $\hat{M}_{ab} = \hat{M}_{ba}$ has been used in the next-to-last term. According to Eq. (65), the term $\Delta \mathbf{x}_{1\text{PN}}$ is separated into a monopole and quadrupole term, $\Delta \mathbf{x}_{1\text{PN}} = \Delta \mathbf{x}_{1\text{PN}}^M + \Delta \mathbf{x}_{1\text{PN}}^Q$, given by Eq. (66) and (67), respectively.

3. The term $\ddot{x}_C^i$

Using (46) one obtains

$$\frac{\ddot{x}_C^i}{c^2} = -2 \frac{GM}{c^2} \frac{x_N^i}{(x_N)^3} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}}{c} + 6 \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^5} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}}{c} - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^7} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}}{c}. \quad (\text{E19})$$

According to Eq. (61), the term $\Delta \dot{\mathbf{x}}_{1\text{PN}}$ is separated into a monopole and quadrupole term, $\Delta \dot{\mathbf{x}}_{1\text{PN}} = \Delta \dot{\mathbf{x}}_{1\text{PN}}^M + \Delta \dot{\mathbf{x}}_{1\text{PN}}^Q$, given by Eq. (62) and (63), respectively.

4. The term $\ddot{x}_D^i$

Using (46) one obtains

$$\frac{\ddot{x}_D^i}{c^2} = +4 \frac{GM}{c^2} \sigma^i \frac{1}{(x_N)^3} \frac{\mathbf{x}_N \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}}{c} - 12 \frac{G\hat{M}_{aj}}{c^2} \sigma^i \frac{x_N^a}{(x_N)^5} \frac{\Delta \dot{x}_{1\text{PN}}^j}{c} + 30 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{x_N^a x_N^b}{(x_N)^7} \frac{\mathbf{x}_N \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}}{c}. \quad (\text{E20})$$

According to Eq. (61), the term $\Delta \dot{\mathbf{x}}_{1\text{PN}}$ is separated into a monopole and quadrupole term, $\Delta \dot{\mathbf{x}}_{1\text{PN}} = \Delta \dot{\mathbf{x}}_{1\text{PN}}^M + \Delta \dot{\mathbf{x}}_{1\text{PN}}^Q$, given by Eq. (62) and (63), respectively.

5. The term $\ddot{x}_E^i$

Using (46) one obtains

$$\frac{\ddot{x}_E^i}{c^2} = +4 \frac{GM}{c^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} \frac{\Delta \dot{x}_{1\text{PN}}^i}{c} - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \frac{x_N^a}{(x_N)^5} \frac{\Delta \dot{x}_{1\text{PN}}^i}{c} + 30 \frac{G\hat{M}_{ab}}{c^2} x_N^a x_N^b \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^7} \frac{\Delta \dot{x}_{1\text{PN}}^i}{c}. \quad (\text{E21})$$

According to Eq. (61), the term $\Delta \dot{\mathbf{x}}_{1\text{PN}}$ is separated into a monopole and quadrupole term, $\Delta \dot{\mathbf{x}}_{1\text{PN}} = \Delta \dot{\mathbf{x}}_{1\text{PN}}^M + \Delta \dot{\mathbf{x}}_{1\text{PN}}^Q$, given by Eq. (62) and (63), respectively.

6. The term $\ddot{x}_F^i$

Using (41) and (46) one obtains

$$\begin{aligned} \frac{\ddot{x}_F^i}{c^2} = & +4 \frac{G^2 M^2}{c^4} \frac{x_N^i}{(x_N)^4} - 12 \frac{GM}{c^2} \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^6} + 36 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^8} - 18 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ci}}{c^2} \frac{x_N^a x_N^b x_N^c}{(x_N)^{10}} \\ & + 45 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \frac{x_N^a x_N^b x_N^c x_N^d x_N^i}{(x_N)^{12}}. \end{aligned} \quad (\text{E22})$$

7. The term $\ddot{x}_G^i$

Using (69) by performing the spatial derivatives and then using $x^i = x_N^i + \mathcal{O}(c^{-2})$, one obtains

$$\begin{aligned} \frac{\ddot{x}_G^i}{c^2} = & +2 \frac{G^2 M^2}{c^4} \frac{x_N^i}{(x_N)^4} + 12 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^i}{(x_N)^6} \hat{n}_N^{ab} - 3 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{(\hat{n}_N^{ab})_{,i}}{(x_N)^4} + \frac{9}{5} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^i}{(x_N)^8} \\ & + \frac{54}{7} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \frac{x_N^i}{(x_N)^8} \hat{n}_N^{bc} - \frac{9}{7} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \frac{(\hat{n}_N^{bc})_{,i}}{(x_N)^6} + \frac{27}{2} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \frac{x_N^i}{(x_N)^8} \hat{n}_N^{abcd} \\ & - \frac{9}{4} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \frac{(\hat{n}_N^{abcd})_{,i}}{(x_N)^6}, \end{aligned} \quad (\text{E23})$$

where the STF tensors $\hat{n}_N^{ab}$ and $\hat{n}_N^{abcd}$ are given by were by Eq. (B13) and their spatial derivatives are given by were by Eq. (B14) in B. By using $\hat{M}_{ab} \delta^{ab} = 0$ and summarizing some terms, the expression (E23) can be simplified. Such simplifications will not be performed at this stage but will be postponed for awhile.

8. The term $\ddot{x}_H^i$

Using (69) by performing the spatial derivatives and then using $x^i = x_N^i + \mathcal{O}(c^{-2})$, one obtains

$$\begin{aligned} \frac{\ddot{x}_H^i}{c^2} = & -4 \frac{G^2 M^2}{c^4} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^4} - 24 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^6} \hat{n}_N^{ab} + 6 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^i \sigma^j \frac{(\hat{n}_N^{ab})_{,j}}{(x_N)^4} - \frac{18}{5} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} \\ & - \frac{108}{7} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} \hat{n}_N^{bc} + \frac{18}{7} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \sigma^i \sigma^j \frac{(\hat{n}_N^{bc})_{,j}}{(x_N)^6} - 27 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} \hat{n}_N^{abcd} \\ & + \frac{9}{2} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^i \sigma^j \frac{(\hat{n}_N^{abcd})_{,j}}{(x_N)^6}, \end{aligned} \quad (\text{E24})$$

where $\hat{n}_N^{ab}$ and $\hat{n}_N^{abcd}$ are given by Eq. (B13). By using $\hat{M}_{ab} \delta^{ab} = 0$ and summarizing some terms, the expression (E24) can be simplified. Such simplifications will not be performed at this stage but will be postponed for awhile.

9. The term $\ddot{x}_I^i$

Using (70) by performing the spatial derivatives and then using $x^i = x_N^i + \mathcal{O}(c^{-2})$, one obtains

$$\begin{aligned} \frac{\ddot{x}_I^i}{c^2} = & + \frac{G^2 M^2}{c^4} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^4} + 3 \frac{G^2 M^2}{c^4} \frac{x_N^i}{(x_N)^4} - 4 \frac{G^2 M^2}{c^4} (d_\sigma)^2 \frac{x_N^i}{(x_N)^6} + 4 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^j \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^6} T_N^{ijab} \\ & - \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^j \sigma^k \frac{1}{(x_N)^4} (T_N^{ijab})_{,k} + 6 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^j \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} T_N^{ijabcd} \\ & - \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^j \sigma^k \frac{1}{(x_N)^6} (T_N^{ijabcd})_{,k} + \frac{96}{35} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} - \frac{36}{35} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ai}}{c^2} \sigma^b \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8}, \end{aligned} \quad (\text{E25})$$

where in the first line relation (38) has been used. The tensors T_N^{ijab} and T_N^{ijkl} are abbreviations to simplify equation (E25). These auxiliary quantities are given by Eqs. (C1) and (C3) and their spatial derivatives by Eqs. (C2) and (C4) in C.

10. The term $\ddot{x}_J^i$

Using (70) by performing the spatial derivatives and then using $x^i = x_N^i + \mathcal{O}(c^{-2})$, one obtains

$$\begin{aligned} \ddot{x}_J^i = & + \frac{G^2 M^2}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^4} - 3 \frac{G^2 M^2}{c^4} \frac{x_N^i}{(x_N)^4} + 2 \frac{G^2 M^2}{c^4} (d_\sigma)^2 \frac{x_N^i}{(x_N)^6} - 2 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^j \sigma^k \frac{x_N^i}{(x_N)^6} T_N^{jkab} \\ & + \frac{1}{2} \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^j \sigma^k \frac{1}{(x_N)^4} \left(T_N^{jkab} \right)_{,i} - 3 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^j \sigma^k \frac{x_N^i}{(x_N)^8} T_N^{jkabcd} \\ & + \frac{1}{2} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^j \sigma^k \frac{1}{(x_N)^6} \left(T_N^{jkabcd} \right)_{,i} - \frac{48}{35} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^i}{(x_N)^8} + \frac{18}{35} \frac{G\hat{M}_{aj}}{c^2} \frac{G\hat{M}_{ak}}{c^2} \sigma^j \sigma^k \frac{x_N^i}{(x_N)^8}, \end{aligned} \quad (\text{E26})$$

where the tensors T_N^{jkab} and T_N^{jkabcd} are given by Eqs. (C1) and (C3), while their spatial derivatives are given by Eqs. (C2) and (C4).

11. The terms proportional to $M \times M$

Summarizing all those terms in Eq. (E16) and Eqs. (E18) - (E26) which are proportional to $M \times M$ one obtains:

$$\begin{aligned} \frac{\ddot{x}_{2\text{PN}}^{i \text{ M} \times \text{M}}}{c^2} = & -2 \frac{GM}{c^2} \frac{\Delta x_{1\text{PN}}^i}{(x_N)^3} + 6 \frac{GM}{c^2} \frac{x_N^i}{(x_N)^5} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{M}}) + 4 \frac{GM}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{M}}}{(x_N)^3} - 12 \frac{GM}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^5} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{M}}) \\ & - 2 \frac{GM}{c^2} \frac{x_N^i}{(x_N)^3} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}}{c} + 4 \frac{GM}{c^2} \sigma^i \frac{1}{(x_N)^3} \frac{\mathbf{x}_N \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}}{c} + 4 \frac{GM}{c^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} \frac{\Delta \dot{x}_{1\text{PN}}^i}{c} + 6 \frac{G^2 M^2}{c^4} \frac{x_N^i}{(x_N)^4} \\ & - 4 \frac{G^2 M^2}{c^4} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^4} + 2 \frac{G^2 M^2}{c^4} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^4} - 2 \frac{G^2 M^2}{c^4} (d_\sigma)^2 \frac{x_N^i}{(x_N)^6}, \end{aligned} \quad (\text{E27})$$

where $\Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}$ and $\Delta \mathbf{x}_{1\text{PN}}^{\text{M}}$ are given by Eqs. (62) and (66), respectively. The equation (E27) can be written in the form given by (75) by means of relation (37). Then, the coefficients in (75) read as follows:

$$\mathcal{E}_{(3)}^i = -\frac{4}{(d_\sigma)^2} d_\sigma^i, \quad (\text{E28})$$

$$\mathcal{E}_{(4)}^i = -4 \sigma^i, \quad (\text{E29})$$

$$\mathcal{E}_{(5)}^i = -12 d_\sigma^i - 12 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^i, \quad (\text{E30})$$

$$\mathcal{E}_{(6)}^i = -2 (d_\sigma)^2 \sigma^i, \quad (\text{E31})$$

$$\mathcal{F}_{(2)}^i = -\frac{4}{(d_\sigma)^2} d_\sigma^i, \quad (\text{E32})$$

$$\mathcal{F}_{(3)}^i = +4 \sigma^i - \frac{4}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^i, \quad (\text{E33})$$

$$\mathcal{F}_{(4)}^i = +6 d_\sigma^i, \quad (\text{E34})$$

$$\mathcal{F}_{(5)}^i = -12 (d_\sigma)^2 \sigma^i + 12 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^i, \quad (\text{E35})$$

$$\mathcal{F}_{(6)}^i = -2 (d_\sigma)^2 d_\sigma^i, \quad (\text{E36})$$

$$\mathcal{G}_{(5)}^i = +12 d_\sigma^i, \quad (\text{E37})$$

$$\mathcal{H}_{(3)}^i = -8 \sigma^i, \quad (\text{E38})$$

$$\mathcal{H}_{(5)}^i = +12 (d_\sigma)^2 \sigma^i. \quad (\text{E39})$$

12. The terms proportional to $M \times \hat{M}_{ab}$

Summarizing all those terms in Eq. (E16) and Eqs. (E18) - (E26) which are proportional to $M \times \hat{M}_{ab}$ one obtains

$$\begin{aligned}
\frac{\ddot{x}_{2\text{PN}}^{i\text{M}\times\text{Q}}}{c^2} = & -2 \frac{GM}{c^2} \frac{\Delta x_{1\text{PN}}^{i\text{Q}}}{(x_N)^3} + 6 \frac{GM}{c^2} \frac{x_N^i}{(x_N)^5} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{Q}}) + 6 \frac{G\hat{M}_{ai}}{c^2} \frac{\Delta x_{1\text{PN}}^{a\text{M}}}{(x_N)^5} \\
& - 30 \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^7} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{M}}) - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b}{(x_N)^7} \Delta x_{1\text{PN}}^{i\text{M}} - 30 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^i}{(x_N)^7} \Delta x_{1\text{PN}}^{b\text{M}} \\
& + 105 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^9} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{M}}) + 4 \frac{GM}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{Q}}}{(x_N)^3} \\
& - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \sigma^i \frac{\Delta x_{1\text{PN}}^{a\text{M}}}{(x_N)^5} - 12 \frac{GM}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^5} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{Q}}) \\
& + 60 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \sigma^i \frac{x_N^a}{(x_N)^7} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{M}}) + 30 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{x_N^a x_N^b}{(x_N)^9} (\boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{M}}) \\
& + 60 \frac{G\hat{M}_{ab}}{c^2} \sigma^i x_N^a \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^7} \Delta x_{1\text{PN}}^{b\text{M}} - 210 \frac{G\hat{M}_{ab}}{c^2} \sigma^i x_N^a x_N^b \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^9} (\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{M}}) \\
& - 2 \frac{GM}{c^2} \frac{x_N^i}{(x_N)^3} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}}{c} + 6 \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^5} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}}{c} - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^7} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}}{c} \\
& + 4 \frac{GM}{c^2} \sigma^i \frac{1}{(x_N)^3} \frac{\mathbf{x}_N \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}}{c} - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{x_N^a}{(x_N)^5} \frac{\Delta \dot{x}_{1\text{PN}}^{b\text{M}}}{c} + 4 \frac{GM}{c^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^3} \frac{\Delta \dot{x}_{1\text{PN}}^{i\text{Q}}}{c} \\
& + 30 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{x_N^a x_N^b}{(x_N)^7} \frac{\mathbf{x}_N \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}}{c} - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \frac{x_N^a}{(x_N)^5} \frac{\Delta \dot{x}_{1\text{PN}}^{i\text{M}}}{c} - 12 \frac{GM}{c^2} \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^6} \\
& + 30 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b}{(x_N)^7} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^7} \frac{\Delta \dot{x}_{1\text{PN}}^{i\text{M}}}{c} + 36 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^8} + 12 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^i}{(x_N)^6} \hat{n}_N^{ab} \\
& - 3 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{(\hat{n}_N^{ab})_{,i}}{(x_N)^4} - 24 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^6} \hat{n}_N^{ab} + 6 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^i \sigma^j \frac{(\hat{n}_N^{ab})_{,j}}{(x_N)^4} \\
& + 4 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^j \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^6} T_N^{ijab} - \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^j \sigma^k \frac{1}{(x_N)^4} (T_N^{ijab})_{,k} \\
& - 2 \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^j \sigma^k \frac{x_N^i}{(x_N)^6} T_N^{jkab} + \frac{1}{2} \frac{GM}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^j \sigma^k \frac{1}{(x_N)^4} (T_N^{jkab})_{,i} .
\end{aligned} \tag{E40}$$

One may rewrite Eq. (E40) in the form as given by Eq. (78) by inserting the explicit form of $\Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}$, $\Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}$ and $\Delta \mathbf{x}_{1\text{PN}}^{\text{M}}$, $\Delta \mathbf{x}_{1\text{PN}}^{\text{Q}}$, as given by Eqs. (62), (63) and Eqs. (66), (67), respectively. Furthermore, one has to insert the explicit expressions for T_N^{ijab} and its spatial derivative as given by Eqs. (C1) and (C2). In addition, relations (37) - (39) are used several times. Then, one finds the following time-independent tensorial coefficients in Eq. (78):

$$\mathcal{K}_{(3)}^{iab} = + \frac{8}{(d_\sigma)^4} d_\sigma^a \delta^{bi} - \frac{4}{(d_\sigma)^4} \sigma^a \sigma^b d_\sigma^i - \frac{8}{(d_\sigma)^4} \sigma^a d_\sigma^b \sigma^i - \frac{16}{(d_\sigma)^6} d_\sigma^a d_\sigma^b d_\sigma^i . \tag{E41}$$

$$\mathcal{K}_{(4)}^{iab} = + \frac{4}{(d_\sigma)^2} \sigma^a \delta^{bi} + \frac{4}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^i - \frac{8}{(d_\sigma)^4} \sigma^a d_\sigma^b d_\sigma^i + \frac{16}{(d_\sigma)^4} d_\sigma^a d_\sigma^b \sigma^i . \tag{E42}$$

$$\begin{aligned}
\mathcal{K}_{(5)}^{iab} = & -\frac{12}{(d_\sigma)^2} d_\sigma^a \delta^{bi} - \frac{18}{(d_\sigma)^2} \sigma^a \sigma^b d_\sigma^i - \frac{12}{(d_\sigma)^2} \sigma^a d_\sigma^b \sigma^i - \frac{24}{(d_\sigma)^4} d_\sigma^a d_\sigma^b d_\sigma^i + \frac{6}{(d_\sigma)^2} \frac{\sigma \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b d_\sigma^i \\
& + \frac{12}{(d_\sigma)^2} \frac{\sigma \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b \sigma^i + \frac{12}{(d_\sigma)^4} \frac{\sigma \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b d_\sigma^i - 6 \frac{\sigma \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b d_\sigma^i + 12 \frac{\sigma \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b \sigma^i \\
& + \frac{6}{(d_\sigma)^2} \frac{\sigma \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^i + \frac{6}{x_0} \sigma^a \sigma^b \sigma^i + \frac{12}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a d_\sigma^b \sigma^i - \frac{6}{(x_0)^3} (d_\sigma)^2 \sigma^a \sigma^b \sigma^i - \frac{12}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^i \\
& + \frac{6}{(x_0)^3} d_\sigma^a d_\sigma^b \sigma^i - \frac{12}{(d_\sigma)^2} (x_0 + \sigma \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^i - \frac{24}{(d_\sigma)^4} (x_0 + \sigma \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b \sigma^i .
\end{aligned} \tag{E43}$$

$$\mathcal{K}_{(6)}^{iab} = +22 \sigma^a \delta^{bi} + \frac{32}{(d_\sigma)^2} \sigma^a d_\sigma^b d_\sigma^i - \frac{64}{(d_\sigma)^2} d_\sigma^a d_\sigma^b \sigma^i . \tag{E44}$$

$$\begin{aligned}
\mathcal{K}_{(7)}^{iab} = & +60 d_\sigma^a \delta^{bi} - 180 \sigma^a \sigma^b d_\sigma^i + 240 \sigma^a d_\sigma^b \sigma^i + \frac{30}{(d_\sigma)^2} d_\sigma^a d_\sigma^b d_\sigma^i - 60 (x_0 + \sigma \cdot \mathbf{x}_0) \sigma^a \delta^{bi} \\
& - 90 (x_0 + \sigma \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^i - \frac{120}{(d_\sigma)^2} (x_0 + \sigma \cdot \mathbf{x}_0) \sigma^a d_\sigma^b d_\sigma^i + \frac{60}{(d_\sigma)^2} (x_0 + \sigma \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b \sigma^i .
\end{aligned} \tag{E45}$$

$$\mathcal{K}_{(8)}^{iab} = +\frac{21}{2} (d_\sigma)^2 \sigma^a \delta^{bi} - 84 (d_\sigma)^2 \sigma^a \sigma^b \sigma^i - 63 \sigma^a d_\sigma^b d_\sigma^i + 69 d_\sigma^a d_\sigma^b \sigma^i . \tag{E46}$$

$$\begin{aligned}
\mathcal{K}_{(9)}^{iab} = & +210 (d_\sigma)^2 \sigma^a \sigma^b d_\sigma^i - 420 (d_\sigma)^2 \sigma^a d_\sigma^b \sigma^i - 210 d_\sigma^a d_\sigma^b d_\sigma^i + 210 (d_\sigma)^2 (x_0 + \sigma \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^i \\
& + 420 (x_0 + \sigma \cdot \mathbf{x}_0) \sigma^a d_\sigma^b d_\sigma^i - 210 (x_0 + \sigma \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b \sigma^i .
\end{aligned} \tag{E47}$$

$$\mathcal{K}_{(10)}^{iab} = +30 (d_\sigma)^4 \sigma^a \sigma^b \sigma^i - 60 (d_\sigma)^2 \sigma^a d_\sigma^b d_\sigma^i - 30 (d_\sigma)^2 d_\sigma^a d_\sigma^b \sigma^i . \tag{E48}$$

$$\mathcal{L}_{(2)}^{iab} = +\frac{8}{(d_\sigma)^4} d_\sigma^a \delta^{bi} - \frac{4}{(d_\sigma)^4} \sigma^a \sigma^b d_\sigma^i - \frac{8}{(d_\sigma)^4} \sigma^a d_\sigma^b \sigma^i - \frac{16}{(d_\sigma)^6} d_\sigma^a d_\sigma^b d_\sigma^i . \tag{E49}$$

$$\begin{aligned}
\mathcal{L}_{(3)}^{iab} = & +\frac{4}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^i + \frac{8}{(d_\sigma)^4} d_\sigma^a d_\sigma^b \sigma^i - \frac{4}{(d_\sigma)^2} \frac{\sigma \cdot \mathbf{x}_0}{x_0} \sigma^a \delta^{bi} + \frac{8}{(d_\sigma)^4} \frac{\sigma \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b d_\sigma^i \\
& - \frac{8}{(d_\sigma)^4} \frac{\sigma \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b \sigma^i + 4 \frac{\sigma \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^i + \frac{4}{(d_\sigma)^2} \frac{\sigma \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^i \\
& - \frac{4}{(d_\sigma)^2} \frac{\sigma \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b \sigma^i - \frac{4}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a \delta^{bi} + \frac{2}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a \sigma^b d_\sigma^i + \frac{4}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a d_\sigma^b \sigma^i \\
& + \frac{8}{(d_\sigma)^4} \frac{1}{x_0} d_\sigma^a d_\sigma^b d_\sigma^i - \frac{2}{(x_0)^3} \sigma^a \sigma^b d_\sigma^i + \frac{8}{(x_0)^3} \sigma^a d_\sigma^b \sigma^i + \frac{2}{(d_\sigma)^2} \frac{1}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^i \\
& + \frac{8}{(d_\sigma)^4} (x_0 + \sigma \cdot \mathbf{x}_0) d_\sigma^a \delta^{bi} - \frac{4}{(d_\sigma)^4} (x_0 + \sigma \cdot \mathbf{x}_0) \sigma^a \sigma^b d_\sigma^i \\
& - \frac{8}{(d_\sigma)^4} (x_0 + \sigma \cdot \mathbf{x}_0) \sigma^a d_\sigma^b \sigma^i - \frac{16}{(d_\sigma)^6} (x_0 + \sigma \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^i .
\end{aligned} \tag{E50}$$

$$\mathcal{L}_{(4)}^{iab} = -\frac{16}{(d_\sigma)^2} d_\sigma^a \delta^{bi} - \frac{22}{(d_\sigma)^2} \sigma^a \sigma^b d_\sigma^i - \frac{20}{(d_\sigma)^2} \sigma^a d_\sigma^b \sigma^i - \frac{28}{(d_\sigma)^4} d_\sigma^a d_\sigma^b d_\sigma^i . \tag{E51}$$

$$\begin{aligned}
\mathcal{L}_{(5)}^{iab} = & +60 \sigma^a \delta^{bi} + 18 \sigma^a \sigma^b \sigma^i + \frac{24}{(d_\sigma)^2} \sigma^a d_\sigma^b d_\sigma^i - \frac{60}{(d_\sigma)^2} d_\sigma^a d_\sigma^b \sigma^i + 6 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^i \\
& - \frac{12}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b d_\sigma^i + \frac{12}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b \sigma^i - 6 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} (d_\sigma)^2 \sigma^a \sigma^b \sigma^i \\
& - 12 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^i + 6 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b \sigma^i - \frac{12}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a d_\sigma^b d_\sigma^i - \frac{6}{x_0} \sigma^a \sigma^b d_\sigma^i \\
& + 6 \frac{(d_\sigma)^2}{(x_0)^3} \sigma^a \sigma^b d_\sigma^i - 12 \frac{(d_\sigma)^2}{(x_0)^3} \sigma^a d_\sigma^b \sigma^i - \frac{6}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^i + \frac{12}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a \delta^{bi} \\
& - \frac{18}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b d_\sigma^i + \frac{36}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b \sigma^i + \frac{24}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^i . \quad (\text{E52})
\end{aligned}$$

$$\mathcal{L}_{(6)}^{iab} = 18 d_\sigma^a \delta^{bi} - 20 \sigma^a \sigma^b d_\sigma^i + 132 \sigma^a d_\sigma^b \sigma^i + \frac{50}{(d_\sigma)^2} d_\sigma^a d_\sigma^b d_\sigma^i . \quad (\text{E53})$$

$$\begin{aligned}
\mathcal{L}_{(7)}^{iab} = & -60 (d_\sigma)^2 \sigma^a \delta^{bi} - 240 (d_\sigma)^2 \sigma^a \sigma^b \sigma^i - 420 \sigma^a d_\sigma^b d_\sigma^i + 210 d_\sigma^a d_\sigma^b \sigma^i - 60 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a \delta^{bi} \\
& + 240 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b d_\sigma^i - 360 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b \sigma^i - \frac{90}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^i . \quad (\text{E54})
\end{aligned}$$

$$\mathcal{L}_{(8)}^{iab} = +\frac{21}{2} (d_\sigma)^2 d_\sigma^a \delta^{bi} + 9 (d_\sigma)^2 \sigma^a \sigma^b d_\sigma^i - 183 (d_\sigma)^2 \sigma^a d_\sigma^b \sigma^i - 24 d_\sigma^a d_\sigma^b d_\sigma^i . \quad (\text{E55})$$

$$\begin{aligned}
\mathcal{L}_{(9)}^{iab} = & +210 (d_\sigma)^4 \sigma^a \sigma^b \sigma^i + 420 (d_\sigma)^2 \sigma^a d_\sigma^b d_\sigma^i - 210 (d_\sigma)^2 d_\sigma^a d_\sigma^b \sigma^i - 210 (d_\sigma)^2 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b d_\sigma^i \\
& + 420 (d_\sigma)^2 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b \sigma^i + 210 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^i . \quad (\text{E56})
\end{aligned}$$

$$\mathcal{L}_{(10)}^{iab} = +30 (d_\sigma)^4 \sigma^a \sigma^b d_\sigma^i + 60 (d_\sigma)^4 \sigma^a d_\sigma^b \sigma^i - 30 (d_\sigma)^2 d_\sigma^a d_\sigma^b d_\sigma^i . \quad (\text{E57})$$

$$\mathcal{M}_{(7)}^{iab} = -60 d_\sigma^a \delta^{bi} + 150 \sigma^a \sigma^b d_\sigma^i - 180 \sigma^a d_\sigma^b \sigma^i . \quad (\text{E58})$$

$$\mathcal{M}_{(8)}^{iab} = 0 . \quad (\text{E59})$$

$$\mathcal{M}_{(9)}^{iab} = -210 (d_\sigma)^2 \sigma^a \sigma^b d_\sigma^i + 420 (d_\sigma)^2 \sigma^a d_\sigma^b \sigma^i + 210 d_\sigma^a d_\sigma^b d_\sigma^i . \quad (\text{E60})$$

$$\mathcal{N}_{(5)}^{iab} = -48 \sigma^a \delta^{bi} - 24 \sigma^a \sigma^b \sigma^i . \quad (\text{E61})$$

$$\mathcal{N}_{(6)}^{iab} = 0 . \quad (\text{E62})$$

$$\mathcal{N}_{(7)}^{iab} = +60 (d_\sigma)^2 \sigma^a \delta^{bi} + 210 (d_\sigma)^2 \sigma^a \sigma^b \sigma^i + 360 \sigma^a d_\sigma^b d_\sigma^i - 180 d_\sigma^a d_\sigma^b \sigma^i . \quad (\text{E63})$$

$$\mathcal{N}_{(8)}^{iab} = 0 . \quad (\text{E64})$$

$$\mathcal{N}_{(9)}^{iab} = -210 (d_\sigma)^4 \sigma^a \sigma^b \sigma^i - 420 (d_\sigma)^2 \sigma^a d_\sigma^b d_\sigma^i + 210 (d_\sigma)^2 d_\sigma^a d_\sigma^b \sigma^i . \quad (\text{E65})$$

13. The terms proportional to $\hat{M}_{ab} \times \hat{M}_{cd}$

Summarizing all those terms in Eq. (E16) and Eqs. (E18) - (E26) which are proportional to $\hat{M}_{ab} \times \hat{M}_{cd}$ one obtains:

$$\begin{aligned}
\frac{\ddot{x}_{2\text{PN}}^{i\text{Q} \times \text{Q}}}{c^2} = & 6 \frac{G\hat{M}_{ai}}{c^2} \frac{\Delta x_{1\text{PN}}^a \text{Q}}{(x_N)^5} - 30 \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^7} \left(\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{Q}} \right) - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b}{(x_N)^7} \Delta x_{1\text{PN}}^{i\text{Q}} \\
& - 30 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^i}{(x_N)^7} \Delta x_{1\text{PN}}^{b\text{Q}} + 105 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^9} \left(\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{Q}} \right) - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \sigma^i \frac{\Delta x_{1\text{PN}}^a \text{Q}}{(x_N)^5} \\
& + 60 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \sigma^i \frac{x_N^a}{(x_N)^7} \left(\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{Q}} \right) + 30 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{x_N^a x_N^b}{(x_N)^7} \left(\boldsymbol{\sigma} \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{Q}} \right) \\
& + 60 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^7} x_N^a \Delta x_{1\text{PN}}^{b\text{Q}} - 210 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^9} x_N^a x_N^b \left(\mathbf{x}_N \cdot \Delta \mathbf{x}_{1\text{PN}}^{\text{Q}} \right) \\
& + 6 \frac{G\hat{M}_{ai}}{c^2} \frac{x_N^a}{(x_N)^5} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}}{c} - 15 \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^a x_N^b x_N^i}{(x_N)^7} \frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}}{c} \\
& - 12 \frac{G\hat{M}_{aj}}{c^2} \sigma^i \frac{x_N^a}{(x_N)^5} \frac{\Delta \dot{x}_{1\text{PN}}^{j\text{Q}}}{c} + 30 \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{x_N^a x_N^b}{(x_N)^7} \frac{\mathbf{x}_N \cdot \Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}}{c} \\
& - 12 \frac{G\hat{M}_{ab}}{c^2} \sigma^b \frac{x_N^a}{(x_N)^5} \frac{\Delta \dot{x}_{1\text{PN}}^{i\text{Q}}}{c} + 30 \frac{G\hat{M}_{ab}}{c^2} x_N^a x_N^b \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^7} \frac{\Delta \dot{x}_{1\text{PN}}^{i\text{Q}}}{c} \\
& - 18 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ci}}{c^2} \frac{x_N^a x_N^b x_N^c}{(x_N)^{10}} + 45 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \frac{x_N^a x_N^b x_N^c x_N^d x_N^i}{(x_N)^{12}} \\
& + \frac{9}{5} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^i}{(x_N)^8} + \frac{54}{7} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \frac{x_N^i}{(x_N)^8} \hat{n}_N^{bc} - \frac{9}{7} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \frac{(\hat{n}_N^{bc})_{,i}}{(x_N)^6} \\
& + \frac{27}{2} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \frac{x_N^i}{(x_N)^8} \hat{n}_N^{abcd} - \frac{9}{4} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \frac{(\hat{n}_N^{abcd})_{,i}}{(x_N)^6} \\
& - \frac{18}{5} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} - \frac{108}{7} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} \hat{n}_N^{bc} \\
& + \frac{18}{7} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \sigma^i \sigma^j \frac{(\hat{n}_N^{bc})_{,j}}{(x_N)^6} - 27 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} \hat{n}_N^{abcd} \\
& + \frac{9}{2} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^i \sigma^j \frac{(\hat{n}_N^{abcd})_{,j}}{(x_N)^6} \\
& + 6 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^j \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} T_N^{ijabcd} - \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^j \sigma^k \frac{1}{(x_N)^6} \left(T_N^{ijabcd} \right)_{,k} \\
& + \frac{96}{35} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^i \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} - \frac{36}{35} \frac{G\hat{M}_{ai}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \sigma^b \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{(x_N)^8} \\
& - 3 \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^j \sigma^k \frac{x_N^i}{(x_N)^8} T_N^{jkabcd} + \frac{1}{2} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \sigma^j \sigma^k \frac{1}{(x_N)^6} \left(T_N^{jkabcd} \right)_{,i} \\
& - \frac{48}{35} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ab}}{c^2} \frac{x_N^i}{(x_N)^8} + \frac{18}{35} \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{ac}}{c^2} \sigma^b \sigma^c \frac{x_N^i}{(x_N)^8} .
\end{aligned} \tag{E66}$$

One may rewrite Eq. (E66) in the form as given by Eq. (79) by inserting the explicit form of $\Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{M}}$, $\Delta \dot{\mathbf{x}}_{1\text{PN}}^{\text{Q}}$ and $\Delta \mathbf{x}_{1\text{PN}}^{\text{M}}$, $\Delta \mathbf{x}_{1\text{PN}}^{\text{Q}}$, as given by Eqs. (62), (63) and Eqs. (66), (67), respectively. Furthermore, one has to insert the expression for T_N^{ijkl} and its spatial derivative as given by Eqs. (C3) and (C4). Furthermore, the STF structure of the indices of the quadrupole tensors $\hat{M}_{ab}$ and $\hat{M}_{cd}$ implies that the dummy variables a, b, c, d are symmetric under exchange ($a \leftrightarrow b$) and ($c \leftrightarrow d$) as well as ($a \leftrightarrow c \wedge b \leftrightarrow d$) and ($a \leftrightarrow d \wedge b \leftrightarrow c$), which simplify the calculations significantly. Also the relations (37) - (39) are frequently been used. Then, one finds the following time-independent

tensorial coefficients in Eq. (79):

$$\begin{aligned} \mathcal{P}_{(5)}^{iabcd} = & + \frac{24}{(d_\sigma)^4} d_\sigma^a \delta^{bc} \delta^{di} - \frac{24}{(d_\sigma)^4} \sigma^a \sigma^b d_\sigma^c \delta^{di} - \frac{6}{(d_\sigma)^4} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i - \frac{48}{(d_\sigma)^4} \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i + \frac{24}{(d_\sigma)^4} \sigma^a \delta^{bc} d_\sigma^d \sigma^i \\ & - \frac{48}{(d_\sigma)^6} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} - \frac{24}{(d_\sigma)^6} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{48}{(d_\sigma)^6} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i. \end{aligned} \quad (\text{E67})$$

$$\begin{aligned} \mathcal{P}_{(6)}^{iabcd} = & + \frac{126}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c \delta^{di} - \frac{12}{(d_\sigma)^2} \sigma^a \delta^{bc} \delta^{di} - \frac{36}{(d_\sigma)^2} \sigma^a \delta^{bc} \sigma^d \sigma^i + \frac{48}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i + \frac{96}{(d_\sigma)^4} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} \\ & + \frac{168}{(d_\sigma)^4} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} + \frac{84}{(d_\sigma)^4} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i - \frac{120}{(d_\sigma)^4} \sigma^a \delta^{bc} d_\sigma^d d_\sigma^i + \frac{72}{(d_\sigma)^4} d_\sigma^a \delta^{bc} d_\sigma^d \sigma^i + \frac{48}{(d_\sigma)^4} d_\sigma^a d_\sigma^b \sigma^c \sigma^d \sigma^i \\ & - \frac{72}{(d_\sigma)^4} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i + \frac{96}{(d_\sigma)^6} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{144}{(d_\sigma)^6} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i. \end{aligned} \quad (\text{E68})$$

$$\begin{aligned} \mathcal{P}_{(7)}^{iabcd} = & - \frac{180}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i + \frac{420}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i - \frac{120}{(d_\sigma)^2} \sigma^a \delta^{bc} d_\sigma^d \sigma^i + \frac{180}{(d_\sigma)^4} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} - \frac{270}{(d_\sigma)^4} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\ & - \frac{120}{(d_\sigma)^4} d_\sigma^a \delta^{bc} d_\sigma^d d_\sigma^i + \frac{120}{(d_\sigma)^4} \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i + \frac{540}{(d_\sigma)^4} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{120}{(d_\sigma)^6} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\ & - \frac{30}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b d_\sigma^c \delta^{di} + \frac{135}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i - \frac{60}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \delta^{bc} \sigma^d d_\sigma^i + \frac{60}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a \delta^{bc} \sigma^d \sigma^i \\ & - \frac{60}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} + \frac{150}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i + \frac{240}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i \\ & - \frac{300}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + 60 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b \sigma^c \delta^{di} + 30 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b d_\sigma^c \delta^{di} - 75 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i \\ & + 180 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i - \frac{30}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} + \frac{120}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i + \frac{75}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\ & - \frac{150}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{30}{x_0} \sigma^a \sigma^b \sigma^c \delta^{di} + \frac{45}{x_0} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i + \frac{60}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{60}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} \\ & + \frac{120}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a d_\sigma^b \sigma^c \sigma^d d_\sigma^i - \frac{60}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a \delta^{bc} d_\sigma^d d_\sigma^i + \frac{60}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a \delta^{bc} d_\sigma^d \sigma^i + \frac{60}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a d_\sigma^b \sigma^c \sigma^d \sigma^i \\ & + \frac{240}{(d_\sigma)^4} \frac{1}{x_0} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{120}{(d_\sigma)^4} \frac{1}{x_0} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i - \frac{30}{(x_0)^3} (d_\sigma)^2 \sigma^a \sigma^b \sigma^c \delta^{di} - \frac{45}{(x_0)^3} (d_\sigma)^2 \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i \\ & + \frac{60}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} + \frac{30}{(x_0)^3} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{210}{(x_0)^3} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i + \frac{180}{(x_0)^3} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i \\ & + \frac{75}{(x_0)^3} d_\sigma^a d_\sigma^b \sigma^c \sigma^d \sigma^i + \frac{90}{(d_\sigma)^2} \frac{1}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{60}{(d_\sigma)^2} \frac{1}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i \\ & - \frac{60}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^c \delta^{di} - \frac{90}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i - \frac{120}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b \sigma^c \delta^{di} \\ & + \frac{120}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b d_\sigma^c \delta^{di} - \frac{240}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b \sigma^c \sigma^d d_\sigma^i + \frac{120}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \delta^{bc} d_\sigma^d d_\sigma^i \\ & - \frac{120}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a \delta^{bc} d_\sigma^d \sigma^i - \frac{120}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b \sigma^c \sigma^d \sigma^i - \frac{480}{(d_\sigma)^6} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\ & + \frac{240}{(d_\sigma)^6} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i. \end{aligned} \quad (\text{E69})$$

$$\begin{aligned}
\mathcal{P}_{(8)}^{iabcd} = & +9\sigma^a\delta^{bc}\delta^{di} - \frac{399}{2}\sigma^a\sigma^b\sigma^c\delta^{di} + 3\delta^{ac}\delta^{bd}\sigma^i - 351\sigma^a\sigma^b\sigma^c\sigma^d\sigma^i + 57\sigma^a\delta^{bc}\sigma^d\sigma^i - \frac{66}{(d_\sigma)^2}d_\sigma^ad_\sigma^b\sigma^c\delta^{di} \\
& - \frac{132}{(d_\sigma)^2}\sigma^ad_\sigma^bd_\sigma^c\delta^{di} + \frac{120}{(d_\sigma)^2}d_\sigma^a\delta^{bc}\sigma^d d_\sigma^i - \frac{468}{(d_\sigma)^2}d_\sigma^a\sigma^b\sigma^c\sigma^d d_\sigma^i - \frac{396}{(d_\sigma)^2}\sigma^a\sigma^bd_\sigma^cd_\sigma^d\sigma^i - \frac{216}{(d_\sigma)^2}\sigma^ad_\sigma^b\sigma^cd_\sigma^d\sigma^i \\
& - \frac{84}{(d_\sigma)^2}d_\sigma^a\delta^{bc}d_\sigma^d\sigma^i - \frac{1332}{(d_\sigma)^4}\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d d_\sigma^i + \frac{648}{(d_\sigma)^4}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i. \tag{E70}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{(9)}^{iabcd} = & +210\sigma^a\sigma^b\sigma^c\sigma^d d_\sigma^i - 420\sigma^a\sigma^b\sigma^cd_\sigma^d\sigma^i + \frac{210}{(d_\sigma)^2}d_\sigma^ad_\sigma^b\sigma^c\sigma^d d_\sigma^i - \frac{840}{(d_\sigma)^2}\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i \\
& - \frac{420}{(d_\sigma)^4}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d d_\sigma^i - 105\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{x_0}\sigma^a\sigma^b\sigma^c\sigma^d d_\sigma^i - \frac{105}{(d_\sigma)^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{x_0}d_\sigma^ad_\sigma^b\sigma^c\sigma^d d_\sigma^i \\
& - \frac{420}{(d_\sigma)^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{x_0}\sigma^ad_\sigma^b\sigma^cd_\sigma^d d_\sigma^i + \frac{210}{(d_\sigma)^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{x_0}\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i + \frac{210}{(d_\sigma)^4}\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{x_0}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d d_\sigma^i \\
& + 105(d_\sigma)^2\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{(x_0)^3}\sigma^a\sigma^b\sigma^c\sigma^d d_\sigma^i - 420(d_\sigma)^2\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{(x_0)^3}\sigma^a\sigma^b\sigma^cd_\sigma^d\sigma^i - 210\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{(x_0)^3}\sigma^ad_\sigma^b\sigma^cd_\sigma^d d_\sigma^i \\
& - 420\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{(x_0)^3}\sigma^ad_\sigma^b\sigma^cd_\sigma^d d_\sigma^i + 420\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{(x_0)^3}\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i + \frac{105}{(d_\sigma)^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{x}_0}{(x_0)^3}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d d_\sigma^i \\
& - 105\frac{(d_\sigma)^2}{x_0}\sigma^a\sigma^b\sigma^c\sigma^d\sigma^i - \frac{210}{x_0}\sigma^a\sigma^b\sigma^cd_\sigma^d d_\sigma^i - \frac{105}{x_0}d_\sigma^ad_\sigma^b\sigma^c\sigma^d\sigma^i - \frac{420}{(d_\sigma)^2}\frac{1}{x_0}\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d d_\sigma^i \\
& + \frac{210}{(d_\sigma)^2}\frac{1}{x_0}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i + 105\frac{(d_\sigma)^4}{(x_0)^3}\sigma^a\sigma^b\sigma^c\sigma^d\sigma^i + 420\frac{(d_\sigma)^2}{(x_0)^3}\sigma^a\sigma^b\sigma^cd_\sigma^d d_\sigma^i \\
& - 210\frac{(d_\sigma)^2}{(x_0)^3}\sigma^a\sigma^bd_\sigma^cd_\sigma^d\sigma^i - 420\frac{(d_\sigma)^2}{(x_0)^3}\sigma^ad_\sigma^b\sigma^cd_\sigma^d\sigma^i - \frac{420}{(x_0)^3}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d d_\sigma^i \\
& + \frac{105}{(x_0)^3}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i + 210(x_0 + \boldsymbol{\sigma}\cdot\mathbf{x}_0)\sigma^a\sigma^b\sigma^c\sigma^d\sigma^i \\
& + \frac{420}{(d_\sigma)^2}(x_0 + \boldsymbol{\sigma}\cdot\mathbf{x}_0)\sigma^a\sigma^b\sigma^cd_\sigma^d d_\sigma^i + \frac{210}{(d_\sigma)^2}(x_0 + \boldsymbol{\sigma}\cdot\mathbf{x}_0)d_\sigma^ad_\sigma^b\sigma^c\sigma^d\sigma^i \\
& + \frac{840}{(d_\sigma)^4}(x_0 + \boldsymbol{\sigma}\cdot\mathbf{x}_0)\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i - \frac{420}{(d_\sigma)^4}(x_0 + \boldsymbol{\sigma}\cdot\mathbf{x}_0)d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i. \tag{E71}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{(10)}^{iabcd} = & +\frac{381}{2}(d_\sigma)^2\sigma^a\sigma^b\sigma^c\delta^{di} - 13(d_\sigma)^2\sigma^a\delta^{bc}\delta^{di} + \frac{777}{2}(d_\sigma)^2\sigma^a\sigma^b\sigma^c\sigma^d\sigma^i \\
& + 95(d_\sigma)^2\sigma^a\delta^{bc}\sigma^d\sigma^i - 5(d_\sigma)^2\delta^{ac}\delta^{bd}\sigma^i - 356\sigma^ad_\sigma^bd_\sigma^c\delta^{di} - 172d_\sigma^ad_\sigma^b\sigma^c\delta^{di} \\
& + 1761\sigma^a\sigma^b\sigma^cd_\sigma^d d_\sigma^i - 110\sigma^a\delta^{bc}d_\sigma^d d_\sigma^i - 112d_\sigma^ad_\sigma^b\sigma^c\sigma^d\sigma^i - 218\sigma^ad_\sigma^b\sigma^cd_\sigma^d\sigma^i \\
& - 50d_\sigma^a\delta^{bc}d_\sigma^d\sigma^i + \frac{816}{(d_\sigma)^2}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d d_\sigma^i - \frac{324}{(d_\sigma)^2}d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i. \tag{E72}
\end{aligned}$$

$$\mathcal{P}_{(11)}^{iabcd} = 0. \tag{E73}$$

$$\begin{aligned}
\mathcal{P}_{(12)}^{iabcd} = & -\frac{45}{2}(d_\sigma)^4\sigma^a\sigma^b\sigma^c\delta^{di} - 165(d_\sigma)^4\sigma^a\sigma^b\sigma^c\sigma^d\sigma^i - 75(d_\sigma)^4\sigma^a\delta^{bc}\sigma^d\sigma^i \\
& + 45(d_\sigma)^2\sigma^ad_\sigma^bd_\sigma^c\delta^{di} + \frac{45}{2}(d_\sigma)^2d_\sigma^ad_\sigma^b\sigma^c\delta^{di} - 2235(d_\sigma)^2\sigma^a\sigma^b\sigma^cd_\sigma^d d_\sigma^i \\
& + 150(d_\sigma)^2\sigma^a\delta^{bc}d_\sigma^d d_\sigma^i + 480(d_\sigma)^2\sigma^a\sigma^bd_\sigma^cd_\sigma^d\sigma^i + 960(d_\sigma)^2\sigma^ad_\sigma^b\sigma^cd_\sigma^d\sigma^i \\
& + 75(d_\sigma)^2d_\sigma^a\delta^{bc}d_\sigma^d\sigma^i + 1935d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d d_\sigma^i - 315d_\sigma^ad_\sigma^bd_\sigma^cd_\sigma^d\sigma^i. \tag{E74}
\end{aligned}$$

$$\mathcal{P}_{(13)}^{iabcd} = 0. \quad (\text{E75})$$

$$\begin{aligned} \mathcal{P}_{(14)}^{iabcd} = & -\frac{225}{2} (d_\sigma)^6 \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i + 450 (d_\sigma)^4 \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i + 225 (d_\sigma)^4 \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i \\ & + 450 (d_\sigma)^4 \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i - 450 (d_\sigma)^2 \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{225}{2} (d_\sigma)^2 d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i. \end{aligned} \quad (\text{E76})$$

$$\begin{aligned} \mathcal{Q}_{(4)}^{iabcd} = & +\frac{24}{(d_\sigma)^4} d_\sigma^a \delta^{bc} \delta^{di} - \frac{24}{(d_\sigma)^4} \sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{6}{(d_\sigma)^4} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i - \frac{48}{(d_\sigma)^4} \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i \\ & + \frac{24}{(d_\sigma)^4} \sigma^a \delta^{bc} d_\sigma^d \sigma^i - \frac{48}{(d_\sigma)^6} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} - \frac{24}{(d_\sigma)^6} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{48}{(d_\sigma)^6} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i. \end{aligned} \quad (\text{E77})$$

$$\begin{aligned} \mathcal{Q}_{(5)}^{iabcd} = & +\frac{60}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c \delta^{di} + \frac{30}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i + \frac{120}{(d_\sigma)^4} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} + \frac{72}{(d_\sigma)^4} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} \\ & + \frac{144}{(d_\sigma)^4} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i - \frac{120}{(d_\sigma)^4} \sigma^a \delta^{bc} d_\sigma^d d_\sigma^i + \frac{24}{(d_\sigma)^4} \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{72}{(d_\sigma)^4} d_\sigma^a \delta^{bc} d_\sigma^d \sigma^i \\ & - \frac{144}{(d_\sigma)^4} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i + \frac{96}{(d_\sigma)^6} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{144}{(d_\sigma)^6} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i \\ & + \frac{12}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \delta^{bc} \delta^{di} - \frac{66}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^c \delta^{di} + \frac{36}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \delta^{bc} \sigma^d \sigma^i \\ & - \frac{18}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i - \frac{48}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{24}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} \\ & + \frac{60}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i - \frac{24}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i - \frac{72}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i \\ & + 24 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c \delta^{di} + 12 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i - \frac{24}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} \\ & - \frac{12}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} + \frac{30}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i - \frac{12}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i \\ & - \frac{36}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i - \frac{12}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a d_\sigma^b \sigma^c \delta^{di} + \frac{12}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a \delta^{bc} \delta^{di} \\ & - \frac{36}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a \sigma^b d_\sigma^c \delta^{di} + \frac{15}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i - \frac{24}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i \\ & + \frac{36}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a \delta^{bc} d_\sigma^d \sigma^i - \frac{24}{(d_\sigma)^4} \frac{1}{x_0} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} + \frac{60}{(d_\sigma)^4} \frac{1}{x_0} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\ & - \frac{72}{(d_\sigma)^4} \frac{1}{x_0} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{6}{(x_0)^3} \sigma^a \sigma^b d_\sigma^c \delta^{di} + \frac{48}{(x_0)^3} \sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{15}{(x_0)^3} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i \\ & + \frac{42}{(x_0)^3} \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i - \frac{6}{(d_\sigma)^2} \frac{1}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} + \frac{15}{(d_\sigma)^2} \frac{1}{(x_0)^3} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\ & - \frac{18}{(d_\sigma)^2} \frac{1}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i - \frac{24}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a \delta^{bc} \delta^{di} \\ & + \frac{72}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b d_\sigma^c \delta^{di} + \frac{24}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b \sigma^c \delta^{di} \\ & - \frac{30}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i + \frac{48}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i \\ & - \frac{72}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \delta^{bc} d_\sigma^d \sigma^i + \frac{48}{(d_\sigma)^6} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} \\ & - \frac{120}{(d_\sigma)^6} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i + \frac{144}{(d_\sigma)^6} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i. \end{aligned} \quad (\text{E78})$$

$$\begin{aligned}
\mathcal{Q}_{(6)}^{iabcd} = & + \frac{30}{(d_\sigma)^2} \sigma^a \sigma^b d_\sigma^c \delta^{di} - \frac{12}{(d_\sigma)^2} d_\sigma^a \delta^{bc} \delta^{di} + \frac{12}{(d_\sigma)^2} \sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{312}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i + \frac{60}{(d_\sigma)^2} \sigma^a \delta^{bc} \sigma^d d_\sigma^i \\
& + \frac{444}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i - \frac{192}{(d_\sigma)^2} \sigma^a \delta^{bc} d_\sigma^d \sigma^i + \frac{264}{(d_\sigma)^4} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} - \frac{120}{(d_\sigma)^4} d_\sigma^a \delta^{bc} d_\sigma^d d_\sigma^i \\
& - \frac{408}{(d_\sigma)^4} \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i - \frac{120}{(d_\sigma)^4} \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i + \frac{864}{(d_\sigma)^4} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{120}{(d_\sigma)^6} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i . \tag{E79}
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_{(7)}^{iabcd} = & -60 \sigma^a \sigma^b \sigma^c \delta^{di} - 240 \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i - \frac{120}{(d_\sigma)^2} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} - \frac{120}{(d_\sigma)^2} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} \\
& - \frac{540}{(d_\sigma)^2} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i + \frac{120}{(d_\sigma)^2} \sigma^a \delta^{bc} d_\sigma^d d_\sigma^i - \frac{120}{(d_\sigma)^2} d_\sigma^a \delta^{bc} d_\sigma^d \sigma^i + \frac{240}{(d_\sigma)^2} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i \\
& - \frac{270}{(d_\sigma)^2} \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i - \frac{840}{(d_\sigma)^4} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i + \frac{540}{(d_\sigma)^4} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + 60 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^c \delta^{di} \\
& + 135 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i - 60 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \delta^{bc} \sigma^d \sigma^i + \frac{60}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} \\
& + \frac{30}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{30}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i - \frac{60}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a \delta^{bc} \sigma^d d_\sigma^i \\
& + \frac{150}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b \sigma^c \sigma^d \sigma^i + \frac{420}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i + \frac{540}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\
& - \frac{180}{(d_\sigma)^4} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i - 30 (d_\sigma)^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c \delta^{di} - 105 (d_\sigma)^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i \\
& + 60 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^c \delta^{di} + 30 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b \sigma^c \delta^{di} - 420 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i \\
& + 195 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i + 360 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i + \frac{270}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\
& - \frac{90}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{60}{x_0} \sigma^a \sigma^b d_\sigma^c \delta^{di} - \frac{120}{x_0} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i + \frac{210}{x_0} \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i \\
& - \frac{60}{x_0} \sigma^a \delta^{bc} d_\sigma^d \sigma^i + \frac{30}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} - \frac{225}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{60}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a \delta^{bc} d_\sigma^d d_\sigma^i \\
& + \frac{60}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i + \frac{450}{(d_\sigma)^2} \frac{1}{x_0} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{180}{(d_\sigma)^4} \frac{1}{x_0} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\
& - \frac{30}{(x_0)^3} (d_\sigma)^2 \sigma^a \sigma^b d_\sigma^c \delta^{di} - \frac{60}{(x_0)^3} (d_\sigma)^2 \sigma^a d_\sigma^b \sigma^c \delta^{di} + \frac{120}{(x_0)^3} (d_\sigma)^2 \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i \\
& - \frac{390}{(x_0)^3} (d_\sigma)^2 \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i + \frac{30}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} - \frac{165}{(x_0)^3} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\
& - \frac{360}{(x_0)^3} \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i + \frac{360}{(x_0)^3} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{45}{(d_\sigma)^2} \frac{1}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\
& - \frac{120}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b d_\sigma^c \delta^{di} + \frac{240}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i \\
& + \frac{120}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a \delta^{bc} d_\sigma^d d_\sigma^i - \frac{420}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i \\
& + \frac{120}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \delta^{bc} d_\sigma^d \sigma^i - \frac{60}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} \\
& + \frac{450}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{120}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i \\
& - \frac{900}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^c \sigma^d \sigma^i - \frac{360}{(d_\sigma)^6} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i . \tag{E80}
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_{(8)}^{iabcd} = & +12 d_\sigma^a \delta^{bc} \delta^{di} - 306 \sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{315}{2} \sigma^a \sigma^b d_\sigma^c \delta^{di} - 102 \sigma^a \delta^{bc} \sigma^d d_\sigma^i + \frac{9}{2} \delta^{ac} \delta^{bd} d_\sigma^i \\
& + \frac{1683}{2} \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i - 876 \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i + 93 d_\sigma^a \delta^{bc} \sigma^d \sigma^i - \frac{114}{(d_\sigma)^2} d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} \\
& + \frac{588}{(d_\sigma)^2} \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i + \frac{420}{(d_\sigma)^2} \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i + \frac{60}{(d_\sigma)^2} d_\sigma^a \delta^{bc} d_\sigma^d d_\sigma^i - \frac{1884}{(d_\sigma)^2} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i \\
& - \frac{690}{(d_\sigma)^4} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i .
\end{aligned} \tag{E81}$$

$$\begin{aligned}
\mathcal{Q}_{(9)}^{iabcd} = & +210 (d_\sigma)^2 \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i + 420 \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i + 210 d_\sigma^a d_\sigma^b \sigma^c \sigma^d \sigma^i + \frac{840}{(d_\sigma)^2} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\
& - \frac{420}{(d_\sigma)^2} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i - 105 (d_\sigma)^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i - 105 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i \\
& - 420 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i - \frac{630}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i + \frac{210}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i \\
& + 105 (d_\sigma)^4 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c \sigma^d \sigma^i + 420 (d_\sigma)^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b \sigma^c d_\sigma^d d_\sigma^i - 210 (d_\sigma)^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a \sigma^b d_\sigma^c d_\sigma^d \sigma^i \\
& - 420 (d_\sigma)^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} \sigma^a d_\sigma^b \sigma^c d_\sigma^d \sigma^i - 420 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c \sigma^d d_\sigma^i + 105 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i \\
& + \frac{105}{x_0} (d_\sigma)^2 \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i - \frac{210}{x_0} (d_\sigma)^2 \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i + \frac{105}{x_0} d_\sigma^a d_\sigma^b \sigma^c \sigma^d d_\sigma^i - \frac{420}{x_0} \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i \\
& - \frac{210}{(d_\sigma)^2} \frac{1}{x_0} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - \frac{105}{(x_0)^3} (d_\sigma)^4 \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i + \frac{420}{(x_0)^3} (d_\sigma)^4 \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i \\
& + \frac{210}{(x_0)^3} (d_\sigma)^2 d_\sigma^a d_\sigma^b \sigma^c \sigma^d d_\sigma^i + \frac{420}{(x_0)^3} (d_\sigma)^2 \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i - \frac{420}{(x_0)^3} (d_\sigma)^2 \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i \\
& - \frac{105}{(x_0)^3} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - 210 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i + 420 (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i \\
& - \frac{210}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b \sigma^c \sigma^d d_\sigma^i + \frac{840}{(d_\sigma)^2} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i \\
& + \frac{420}{(d_\sigma)^4} (x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0) d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i .
\end{aligned} \tag{E82}$$

$$\begin{aligned}
\mathcal{Q}_{(10)}^{iabcd} = & +385 (d_\sigma)^2 \sigma^a d_\sigma^b \sigma^c \delta^{di} - 13 (d_\sigma)^2 d_\sigma^a \delta^{bc} \delta^{di} + \frac{397}{2} (d_\sigma)^2 \sigma^a \sigma^b d_\sigma^c \delta^{di} \\
& - 5 (d_\sigma)^2 \delta^{ac} \delta^{bd} d_\sigma^i + 115 (d_\sigma)^2 \sigma^a \delta^{bc} \sigma^d d_\sigma^i - \frac{2127}{2} (d_\sigma)^2 \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i \\
& + 220 (d_\sigma)^2 \sigma^a \delta^{bc} d_\sigma^d \sigma^i + 1371 (d_\sigma)^2 \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i - 180 d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} + 716 \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\
& + 1438 \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i - 70 d_\sigma^a \delta^{bc} d_\sigma^d d_\sigma^i - 96 d_\sigma^a d_\sigma^b d_\sigma^c \sigma^d \sigma^i + \frac{300}{(d_\sigma)^2} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i .
\end{aligned} \tag{E83}$$

$$\mathcal{Q}_{(11)}^{iabcd} = 0 . \tag{E84}$$

$$\begin{aligned}
\mathcal{Q}_{(12)}^{iabcd} = & -45 (d_\sigma)^4 \sigma^a d_\sigma^b \sigma^c \delta^{di} - \frac{45}{2} (d_\sigma)^4 \sigma^a \sigma^b d_\sigma^c \delta^{di} + \frac{1305}{2} (d_\sigma)^4 \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i \\
& - 75 (d_\sigma)^4 \sigma^a \delta^{bc} \sigma^d d_\sigma^i - 585 (d_\sigma)^4 \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i - 150 (d_\sigma)^4 \sigma^a \delta^{bc} d_\sigma^d \sigma^i \\
& + \frac{45}{2} (d_\sigma)^2 d_\sigma^a d_\sigma^b d_\sigma^c \delta^{di} - 1155 (d_\sigma)^2 \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i - 2310 (d_\sigma)^2 \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i \\
& + 75 (d_\sigma)^2 d_\sigma^a \delta^{bc} d_\sigma^d d_\sigma^i + 885 (d_\sigma)^2 \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i + \frac{1005}{2} d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i .
\end{aligned} \tag{E85}$$

$$\mathcal{Q}_{(13)}^{iabcd} = 0. \quad (\text{E86})$$

$$\begin{aligned} \mathcal{Q}_{(14)}^{iabcd} = & -\frac{225}{2} (d_\sigma)^6 \sigma^a \sigma^b \sigma^c \sigma^d d_\sigma^i - 450 (d_\sigma)^6 \sigma^a \sigma^b \sigma^c d_\sigma^d \sigma^i + 225 (d_\sigma)^4 \sigma^a \sigma^b d_\sigma^c d_\sigma^d d_\sigma^i \\ & + 450 (d_\sigma)^4 \sigma^a d_\sigma^b \sigma^c d_\sigma^d d_\sigma^i + 450 (d_\sigma)^4 \sigma^a d_\sigma^b d_\sigma^c d_\sigma^d \sigma^i - \frac{225}{2} (d_\sigma)^2 d_\sigma^a d_\sigma^b d_\sigma^c d_\sigma^d d_\sigma^i. \end{aligned} \quad (\text{E87})$$

Appendix F: Estimation of the expression in (107)

In this appendix we want to proof the inequality in (107) as instructive example. That means we consider the upper limit of the absolute value $|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{\text{1PN}}^{\text{Q}}|$ where $\boldsymbol{\nu}_{\text{1PN}}^{\text{Q}}$ is given by Eq. (97). Using $\left(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{\text{1PN}}^{\text{Q}}\right)^k = \epsilon^{ijk} \sigma^i \nu_{\text{1PN}}^{j\text{Q}}$ one gets

$$\left(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{\text{1PN}}^{\text{Q}}\right)^k = \frac{G\hat{M}_{ab}}{c^2} \epsilon^{ijk} \sigma^i \left[\sum_{n=5,7} \mathcal{D}_{(n)}^{j\text{ab}} \dot{\mathcal{X}}_{(n)}^\infty \right], \quad (\text{F1})$$

where the time-independent coefficients $D_{(n)}^{j\text{ab}}$ are given by Eqs. (56) and (57). By inserting the limits (D19) into (F1) one obtains

$$\left(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{\text{1PN}}^{\text{Q}}\right)^k = \frac{4}{3} \frac{G\hat{M}_{ab}}{c^2} \frac{\epsilon^{ijk} \sigma^i}{(d_\sigma)^4} \left[\mathcal{D}_{(5)}^{j\text{ab}} + \frac{4}{5} \frac{1}{(d_\sigma)^2} \mathcal{D}_{(7)}^{j\text{ab}} \right]. \quad (\text{F2})$$

The origin of the spatial coordinates x^1, x^2, x^3 is located at the center of mass of the body. As stated in Section VII, we assume the x^3 -axis of coordinate system to be aligned with the symmetry axis $\mathbf{e}_3$ of the massive body. Hence, the quadrupole tensor (105) takes the form

$$\hat{M}_{ab} = M J_2 P^2 \left(\frac{1}{3} \delta_{ab} - \delta_{a3} \delta_{b3} \right). \quad (\text{F3})$$

By inserting (F3) into (F2) and taking into account that

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}| = |\boldsymbol{\sigma} \times (\boldsymbol{\nu} \times \boldsymbol{\sigma})|, \quad (\text{F4})$$

which is valid since $\boldsymbol{\sigma}$ is a unit-vector, and using $\boldsymbol{\sigma} \times (\mathbf{d}_\sigma \times \boldsymbol{\sigma}) = \mathbf{d}_\sigma$ and $\boldsymbol{\sigma} \times (\mathbf{x}_0 \times \boldsymbol{\sigma}) = \mathbf{d}_\sigma$, one obtains

$$\left| \boldsymbol{\sigma} \times \boldsymbol{\nu}_{\text{1PN}}^{\text{Q}} \right| = 4 \frac{GM}{c^2} |J_2| \frac{P^2}{(d_\sigma)^3} \left| F_1 \frac{\mathbf{d}_\sigma}{d_\sigma} + F_2 (\boldsymbol{\sigma} \times (\mathbf{e}_3 \times \boldsymbol{\sigma})) \right|, \quad (\text{F5})$$

where the functions read

$$F_1 = 1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 - 4 \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2}, \quad (\text{F6})$$

$$F_2 = 2 \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)}{d_\sigma}. \quad (\text{F7})$$

For determining the limit (F5) it is necessary to take into account that $\boldsymbol{\sigma} \cdot \mathbf{d}_\sigma = 0$ which restricts the possible values for the angles $\delta(\boldsymbol{\sigma}, \mathbf{e}_3)$ and $\delta(\mathbf{d}_\sigma, \mathbf{e}_3)$ (see also endnote [99]). One obtains for the upper limit

$$\left| \boldsymbol{\sigma} \times \boldsymbol{\nu}_{\text{1PN}}^{\text{Q}} \right| \leq 4 \frac{GM}{c^2} |J_2| \frac{P^2}{(d_\sigma)^3}, \quad (\text{F8})$$

which has also been proven in our article [81] (cf. Eq. (12) *ibid.*). Thus, the validity of relation (107) has been shown.

Appendix G: Estimation of the expression in (108)

In this appendix we want to proof the inequality in (108). That means we consider the upper limit of the absolute value $|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}}|$ where $\boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}}$ is given by Eq. (98). Using $(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}})^k = \epsilon^{ijk} \sigma^i \nu_{2\text{PN}}^{j\text{M}\times\text{M}}$ one gets

$$(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}})^k = \frac{G^2 M^2}{c^4} \epsilon^{ijk} \sigma^i \left[\sum_{n=2}^6 \mathcal{F}_{(n)}^j \dot{\chi}_{(n)}^\infty + \mathcal{G}_{(5)}^j \dot{\chi}_{(5)}^\infty \right], \quad (\text{G1})$$

where the time-independent coefficients $\mathcal{F}_{(n)}^j$ and $\mathcal{G}_{(5)}^j$ are given by Eqs. (E32) - (E36) and (E37), respectively. By inserting the limits (D19) into (G1) one obtains

$$\left[\sum_{n=2}^6 \mathcal{F}_{(n)}^j \dot{\chi}_{(n)}^\infty + \mathcal{G}_{(5)}^j \dot{\chi}_{(5)}^\infty \right] = 2 \frac{\mathcal{F}_{(3)}^j}{(d_\sigma)^2} + \frac{4}{3} \frac{\mathcal{F}_{(5)}^j}{(d_\sigma)^4} + \pi \left(\frac{\mathcal{F}_{(2)}^j}{d_\sigma} + \frac{1}{2} \frac{\mathcal{F}_{(4)}^j}{(d_\sigma)^3} + \frac{3}{8} \frac{\mathcal{F}_{(6)}^j}{(d_\sigma)^5} - \frac{1}{6} \frac{\mathcal{G}_{(5)}^j}{(d_\sigma)^3} \right). \quad (\text{G2})$$

By inserting the coefficients $\mathcal{F}_{(n)}^j$ and $\mathcal{G}_{(5)}^j$ into (G2) one obtains

$$\left[\sum_{n=2}^6 \mathcal{F}_{(n)}^j \dot{\chi}_{(n)}^\infty + \mathcal{G}_{(5)}^j \dot{\chi}_{(5)}^\infty \right] = 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^4} d_\sigma^j - \frac{15}{4} \pi \frac{d_\sigma^j}{(d_\sigma)^3} \quad (\text{G3})$$

where all terms proportional to σ^j have been omitted since they do not contribute to the total light deflection because of $\epsilon^{ijk} \sigma^i \sigma^j = 0$. Inserting (G3) into (G1) yields for the absolute value

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}}| = \frac{G^2 M^2}{c^4} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} - 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|, \quad (\text{G4})$$

which is the asserted relation (108) and agrees with the 2PN terms in Eq. (3.2.44) in [19] as well as with the 2PN terms in Eq. (65) in [79], if one uses the PPN parameters in the limit of general relativity. The upper limit (G4) is valid for any astrometric configuration of source, body, and observer, that means for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ as well as for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \geq 0$. We are mainly interested in astrometric configurations (cf. condition (111))

$$\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0, \quad (\text{G5})$$

that means astrometric configurations where the massive body is located between the light source and the observer. In general, condition (G5) holds for any light source far from the Solar System, like stellar light sources. If condition (G5) is assumed, then the upper limit (G4) simplifies as follows:

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{M}}| = \frac{15}{4} \pi \frac{G^2 M^2}{c^4} \frac{1}{(d_\sigma)^2}, \quad (\text{G6})$$

which is the asserted relation (112).

Appendix H: Estimation of the expression in (109)

In this appendix we want to proof the inequality in (109). That means we consider the upper limit of the absolute value $|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}}|$ where $\boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}}$ is given by Eq. (99). Using $(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}})^k = \epsilon^{ijk} \sigma^i \nu_{2\text{PN}}^{j\text{M}\times\text{Q}}$ one gets

$$(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}})^k = \frac{GM}{c^2} \frac{\hat{M}_{ab}}{c^2} \epsilon^{ijk} \sigma^i \left[\sum_{n=2}^{10} \mathcal{L}_{(n)}^{j\text{ab}} \dot{\chi}_{(n)}^\infty + \sum_{n=7}^9 \mathcal{M}_{(n)}^{j\text{ab}} \dot{\chi}_{(n)}^\infty \right], \quad (\text{H1})$$

where the time-independent coefficients $\mathcal{L}_{(n)}^{j\text{ab}}$ and $\mathcal{M}_{(n)}^{j\text{ab}}$ are given by Eqs. (E49) - (E57) and (E58) - (E60), respectively. By inserting the limits (D19) into (H1) one obtains

$$\begin{aligned} \left[\sum_{n=2}^{10} \mathcal{L}_{(n)}^{j\text{ab}} \dot{\chi}_{(n)}^\infty + \sum_{n=7}^9 \mathcal{M}_{(n)}^{j\text{ab}} \dot{\chi}_{(n)}^\infty \right] &= 2 \frac{\mathcal{L}_{(3)}^{j\text{ab}}}{(d_\sigma)^2} + \frac{4}{3} \frac{\mathcal{L}_{(5)}^{j\text{ab}}}{(d_\sigma)^4} + \frac{16}{15} \frac{\mathcal{L}_{(7)}^{j\text{ab}}}{(d_\sigma)^6} + \frac{32}{35} \frac{\mathcal{L}_{(9)}^{j\text{ab}}}{(d_\sigma)^8} \\ &+ \pi \left(\frac{\mathcal{L}_{(2)}^{j\text{ab}}}{d_\sigma} + \frac{1}{2} \frac{\mathcal{L}_{(4)}^{j\text{ab}}}{(d_\sigma)^3} + \frac{3}{8} \frac{\mathcal{L}_{(6)}^{j\text{ab}}}{(d_\sigma)^5} + \frac{5}{16} \frac{\mathcal{L}_{(8)}^{j\text{ab}}}{(d_\sigma)^7} + \frac{35}{128} \frac{\mathcal{L}_{(10)}^{j\text{ab}}}{(d_\sigma)^9} - \frac{3}{40} \frac{\mathcal{M}_{(7)}^{j\text{ab}}}{(d_\sigma)^5} - \frac{5}{112} \frac{\mathcal{M}_{(9)}^{j\text{ab}}}{(d_\sigma)^7} \right). \end{aligned} \quad (\text{H2})$$

Then, by inserting the time-independent coefficients $\mathcal{L}_{(n)}^{jab}$ and $\mathcal{M}_{(n)}^{jab}$ one obtains

$$\left[\sum_{n=2}^{10} \mathcal{L}_{(n)}^{jab} \dot{\mathcal{X}}_{(n)}^\infty + \sum_{n=7}^9 \mathcal{M}_{(n)}^{jab} \dot{\mathcal{Y}}_{(n)}^\infty \right] = \pi \left(\frac{465}{32} \frac{d_\sigma^a \delta^{bj}}{(d_\sigma)^5} - \frac{855}{64} \frac{\sigma^a \sigma^b d_\sigma^j}{(d_\sigma)^5} - \frac{2325}{64} \frac{d_\sigma^a d_\sigma^b d_\sigma^j}{(d_\sigma)^7} \right) \\ + 16 \frac{\sigma^a \delta^{bj}}{(d_\sigma)^4} - 8 \frac{\sigma \cdot \mathbf{x}_0}{x_0} \frac{\sigma^a \delta^{bj}}{(d_\sigma)^4} - 32 \frac{\sigma^a d_\sigma^b d_\sigma^j}{(d_\sigma)^6} - 8 \frac{\sigma \cdot \mathbf{x}_0}{(x_0)^3} \frac{\sigma^a d_\sigma^b d_\sigma^j}{(d_\sigma)^4} - \frac{8}{x_0} \frac{d_\sigma^a \delta^{bj}}{(d_\sigma)^4} - 32 (x_0 + \sigma \cdot \mathbf{x}_0) \frac{d_\sigma^a \delta^{bj}}{(d_\sigma)^6} \\ - \frac{4}{x_0} \frac{\sigma^a \sigma^b d_\sigma^j}{(d_\sigma)^4} + \frac{4}{(d_\sigma)^2} \frac{\sigma^a \sigma^b d_\sigma^j}{(x_0)^3} + 32 (x_0 + \sigma \cdot \mathbf{x}_0) \frac{\sigma^a \sigma^b d_\sigma^j}{(d_\sigma)^6} - \frac{4}{(x_0)^3} \frac{d_\sigma^a d_\sigma^b d_\sigma^j}{(d_\sigma)^4} + 96 (x_0 + \sigma \cdot \mathbf{x}_0) \frac{d_\sigma^a d_\sigma^b d_\sigma^j}{(d_\sigma)^8}, \quad (\text{H3})$$

where terms proportional to σ^j have been omitted because they do not contribute to the total light deflection because of $\epsilon^{ijk} \sigma^i \sigma^j = 0$. Inserting the STF quadrupole moment (F3) into Eq. (H3) yields for the absolute value of (H1)

$$\left| \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}} \right| = \frac{G^2 M^2}{c^4} \frac{|J_2| P^2}{(d_\sigma)^4} \left| G_1 \frac{\mathbf{d}_\sigma}{d_\sigma} + G_2 \boldsymbol{\sigma} \times (\mathbf{e}_3 \times \boldsymbol{\sigma}) \right|, \quad (\text{H4})$$

where also relation (F4) has been used. The functions in (H4) read

$$G_1 = -\frac{15}{32} \pi \left(25 - \frac{57}{2} (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 - \frac{155}{2} \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} \right) + 32 \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 - 3 \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} \right) \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{d_\sigma} \\ + 32 \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} (\boldsymbol{\sigma} \cdot \mathbf{e}_3) - 4 \frac{d_\sigma}{x_0} \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right) + 8 (d_\sigma)^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} (\boldsymbol{\sigma} \cdot \mathbf{e}_3) \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \\ + 4 \frac{(d_\sigma)^3}{(x_0)^3} \left(\frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right), \quad (\text{H5})$$

$$G_2 = -\frac{465}{32} \pi \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} + 32 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{d_\sigma} \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} - 16 (\boldsymbol{\sigma} \cdot \mathbf{e}_3) + 8 \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} (\boldsymbol{\sigma} \cdot \mathbf{e}_3) + 8 \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{x_0}. \quad (\text{H6})$$

We note the relation

$$\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} = \pm \sqrt{1 - \left(\frac{d_\sigma}{x_0} \right)^2} \quad \text{with} \quad 0 \leq \frac{d_\sigma}{x_0} \leq 1, \quad (\text{H7})$$

which is a useful relation for the estimation of (H4). Then, by using the computer algebra system *Maple* [80], one obtains for astrometric configurations (G5) the following upper limit:

$$\left| \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}} \right| \leq 4 \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{(d_\sigma)^2} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} + 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|, \quad (\text{H8})$$

which is the asserted relation in (109). For the estimation in (H8) one has to take into account that $\boldsymbol{\sigma}$ and $\mathbf{d}_\sigma$ are perpendicular to each other, which reduces the possible areas of the angles $\delta(\boldsymbol{\sigma}, \mathbf{e}_3)$ and $\delta(\mathbf{d}_\sigma, \mathbf{e}_3)$. For instance, one obtains (see endnote [99])

$$|\boldsymbol{\sigma} \cdot \mathbf{e}_3| \left| \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \right| \leq \frac{1}{2}, \quad (\text{H9})$$

and

$$\left| \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 - 3 \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} \right) \frac{\mathbf{d}_\sigma}{d_\sigma} + \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \boldsymbol{\sigma} \times (\mathbf{e}_3 \times \boldsymbol{\sigma}) \right| \leq 1. \quad (\text{H10})$$

The upper limit (H8) is valid for any astrometric configuration of source, body, and observer, that means for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ as well as for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 > 0$. If one considers astrometric configurations with condition (G5) then one obtains the following upper limit:

$$\left| \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{M}\times\text{Q}} \right| \leq 15 \pi \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{(d_\sigma)^4} \quad (\text{H11})$$

which is the asserted relation (113).

Appendix I: Estimation of the expression in (110)

In this appendix we want to proof the inequality in (110). That means we consider the upper limit of the absolute value $|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}}|$ where $\boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}}$ is given by Eq. (100). Using $(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}})^k = \epsilon^{ijk} \sigma^i \nu_{2\text{PN}}^{j\text{Q}\times\text{Q}}$ one gets

$$(\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}})^k = \frac{G\hat{M}_{ab}}{c^2} \frac{G\hat{M}_{cd}}{c^2} \epsilon^{ijk} \sigma^i \left[\sum_{n=4}^{14} \mathcal{Q}_{(n)}^{j\text{abcd}} \dot{\chi}_{(n)}^\infty \right], \quad (\text{I1})$$

where the time-independent coefficients $\mathcal{Q}_{(n)}^{j\text{abcd}}$ are given by Eqs. (E77) - (E87). By inserting the limits (D19) into (I1) one obtains

$$\begin{aligned} \left[\sum_{n=4}^{14} \mathcal{Q}_{(n)}^{j\text{abcd}} \dot{\chi}_{(n)}^\infty \right] &= \frac{4}{3} \frac{\mathcal{Q}_{(5)}^{j\text{abcd}}}{(d_\sigma)^4} + \frac{16}{15} \frac{\mathcal{Q}_{(7)}^{j\text{abcd}}}{(d_\sigma)^6} + \frac{32}{35} \frac{\mathcal{Q}_{(9)}^{j\text{abcd}}}{(d_\sigma)^8} \\ &+ \pi \left(\frac{1}{2} \frac{\mathcal{Q}_{(4)}^{j\text{abcd}}}{(d_\sigma)^3} + \frac{3}{8} \frac{\mathcal{Q}_{(6)}^{j\text{abcd}}}{(d_\sigma)^5} + \frac{5}{16} \frac{\mathcal{Q}_{(8)}^{j\text{abcd}}}{(d_\sigma)^7} + \frac{35}{128} \frac{\mathcal{Q}_{(10)}^{j\text{abcd}}}{(d_\sigma)^9} + \frac{63}{256} \frac{\mathcal{Q}_{(12)}^{j\text{abcd}}}{(d_\sigma)^{11}} + \frac{231}{1024} \frac{\mathcal{Q}_{(14)}^{j\text{abcd}}}{(d_\sigma)^{13}} \right), \quad (\text{I2}) \end{aligned}$$

where $\mathcal{Q}_{(11)}^{j\text{abcd}} = \mathcal{Q}_{(13)}^{j\text{abcd}} = 0$ has been taken into account. Then, by inserting the time-independent coefficients $\mathcal{Q}_{(n)}^{j\text{abcd}}$ as well as the product of two STF quadrupole moments in z-direction (F3)

$$\hat{M}_{ab} \hat{M}_{cd} = M^2 |J_2|^2 P^4 \left(\frac{1}{9} \delta_{ab} \delta_{cd} - \frac{1}{3} \delta_{ab} \delta_{c3} \delta_{d3} - \frac{1}{3} \delta_{a3} \delta_{b3} \delta_{cd} + \delta_{a3} \delta_{b3} \delta_{c3} \delta_{d3} \right) \quad (\text{I3})$$

into Eq. (I2) yields for the absolute value of (I1)

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}}| = \frac{G^2 M^2}{c^4} \frac{|J_2|^2 P^4}{(d_\sigma)^6} \left| H_1 \frac{\mathbf{d}_\sigma}{d_\sigma} + H_2 \boldsymbol{\sigma} \times (\mathbf{e}_3 \times \boldsymbol{\sigma}) \right|, \quad (\text{I4})$$

where relation (F4) has been used. The functions H_1 and H_2 read

$$\begin{aligned} H_1 &= -\frac{5}{128} \pi \left(\frac{263}{3} + \frac{787}{4} \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} + \frac{1323}{8} (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} - \frac{3969}{16} \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^4}{(d_\sigma)^4} - \frac{1265}{4} (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 + \frac{5175}{16} (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^4 \right) \\ &+ 24 \left(1 - 2 (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 + (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^4 \right) \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{d_\sigma} - 48 (\boldsymbol{\sigma} \cdot \mathbf{e}_3) \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right) \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \\ &+ 24 (d_\sigma)^2 \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} (\boldsymbol{\sigma} \cdot \mathbf{e}_3) \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 - 4 \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_2)^2}{(d_\sigma)^2} \right) \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} - 12 \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right)^2 \frac{d_\sigma}{x_0} \\ &+ 12 \left(\frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} + 2 (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} - 4 \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^4}{(d_\sigma)^4} + (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^4 \right) \frac{(d_\sigma)^3}{(x_0)^3}, \quad (\text{I5}) \end{aligned}$$

$$\begin{aligned} H_2 &= \frac{105}{256} \pi \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \left(1 - \frac{21}{2} \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} + \frac{9}{2} (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right) - 16 (\boldsymbol{\sigma} \cdot \mathbf{e}_3) \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right) + 24 (\boldsymbol{\sigma} \cdot \mathbf{e}_3) \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right) \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{x_0} \\ &+ 48 (d_\sigma)^2 (\boldsymbol{\sigma} \cdot \mathbf{e}_3) \frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_0}{(x_0)^3} + 24 \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \left(\frac{(\mathbf{d}_\sigma \cdot \mathbf{e}_3)^2}{(d_\sigma)^2} - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right) \frac{(d_\sigma)^3}{(x_0)^3}. \quad (\text{I6}) \end{aligned}$$

Let us recall relation (H7) which is useful for the estimation of (I4). By using the computer algebra system *Maple* [80], one obtains

$$|\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q}\times\text{Q}}| \leq 2 \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^4}{(d_\sigma)^4} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} + 12 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|, \quad (\text{I7})$$

which is the asserted relation (110). In order to show the upper limit in (I7) it is necessary to take into account that $\boldsymbol{\sigma} \cdot \mathbf{d}_\sigma = 0$ which restricts the possible values for the angles $\delta(\boldsymbol{\sigma}, \mathbf{e}_3)$ and $\delta(\mathbf{d}_\sigma, \mathbf{e}_3)$ (see also endnote [99]).

The upper limit (I7) is valid for any astrometric configuration, that means for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ as well as for $\boldsymbol{\sigma} \cdot \mathbf{x}_0 > 0$. If one considers astrometric configurations with condition (G5) then one obtains the following upper limit:

$$\left| \boldsymbol{\sigma} \times \boldsymbol{\nu}_{2\text{PN}}^{\text{Q} \times \text{Q}} \right| \leq \frac{15}{2} \pi \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^2}{(d_\sigma)^6}, \quad (\text{I8})$$

which is the asserted relation (114).

Appendix J: The expressions for the enhanced terms

The expression of $\mathbf{n}_{1\text{PN}}(\mathbf{x}_\text{N})$ follows by inserting (61) - (63) into (120) and is given by

$$\begin{aligned} n_{1\text{PN}}^i(\mathbf{x}_\text{N}) = & -2 \frac{GM}{c^2} \frac{d_\sigma^i}{(d_\sigma)^2} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_\text{N}}{x_\text{N}} \right) + 2 \frac{G\hat{M}_{ab}}{c^2} \frac{\sigma^a \sigma^b \sigma^i - \sigma^a \delta^{bi}}{(x_\text{N})^3} + 6 \frac{G\hat{M}_{ab}}{c^2} \frac{\sigma^a d_\sigma^b d_\sigma^i}{(x_\text{N})^5} \\ & + 8 \frac{G\hat{M}_{ab}}{c^2} \frac{(d_\sigma)^2 \sigma^a \sigma^b d_\sigma^i - d_\sigma^a d_\sigma^b d_\sigma^i}{(d_\sigma)^2} \left[\frac{1}{(d_\sigma)^4} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_\text{N}}{x_\text{N}} \right) + \frac{1}{2} \frac{1}{(d_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_\text{N}}{(x_\text{N})^3} + \frac{3}{8} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_\text{N}}{(x_\text{N})^5} \right] \\ & + 2 \frac{G\hat{M}_{ab}}{c^2} \frac{2 d_\sigma^a \delta^{bi} - 5 \sigma^a \sigma^b d_\sigma^i - 2 \sigma^a d_\sigma^b \sigma^i}{(d_\sigma)^2} \left[\frac{1}{(d_\sigma)^2} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_\text{N}}{x_\text{N}} \right) + \frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_\text{N}}{(x_\text{N})^3} \right], \end{aligned} \quad (\text{J1})$$

where the impact vector is given by (33), that means

$$\mathbf{d}_\sigma = \boldsymbol{\sigma} \times (\mathbf{x}_\text{N} \times \boldsymbol{\sigma}) \quad \text{and} \quad d_\sigma = |\mathbf{d}_\sigma|. \quad (\text{J2})$$

The expression $\mathbf{n}_{1\text{PN}}(\mathbf{x}_{1\text{PN}})$ in (123) is obtained from (J1) by replacing the argument $\mathbf{x}_\text{N}$ by $\mathbf{x}_{1\text{PN}}$ everywhere. That means

$$\begin{aligned} n_{1\text{PN}}^i(\mathbf{x}_{1\text{PN}}) = & -2 \frac{GM}{c^2} \frac{\hat{d}_\sigma^i}{(\hat{d}_\sigma)^2} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_{1\text{PN}}}{x_{1\text{PN}}} \right) + 2 \frac{G\hat{M}_{ab}}{c^2} \frac{\sigma^a \sigma^b \sigma^i - \sigma^a \delta^{bi}}{(x_{1\text{PN}})^3} + 6 \frac{G\hat{M}_{ab}}{c^2} \frac{\sigma^a \hat{d}_\sigma^b \hat{d}_\sigma^i}{(x_{1\text{PN}})^5} \\ & + 8 \frac{G\hat{M}_{ab}}{c^2} \frac{(\hat{d}_\sigma)^2 \sigma^a \sigma^b \hat{d}_\sigma^i - \hat{d}_\sigma^a \hat{d}_\sigma^b \hat{d}_\sigma^i}{(\hat{d}_\sigma)^2} \left[\frac{1}{(\hat{d}_\sigma)^4} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_{1\text{PN}}}{x_{1\text{PN}}} \right) + \frac{1}{2} \frac{1}{(\hat{d}_\sigma)^2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_{1\text{PN}}}{(x_{1\text{PN}})^3} + \frac{3}{8} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_{1\text{PN}}}{(x_{1\text{PN}})^5} \right] \\ & + 2 \frac{G\hat{M}_{ab}}{c^2} \frac{2 \hat{d}_\sigma^a \delta^{bi} - 5 \sigma^a \sigma^b \hat{d}_\sigma^i - 2 \sigma^a \hat{d}_\sigma^b \sigma^i}{(\hat{d}_\sigma)^2} \left[\frac{1}{(\hat{d}_\sigma)^2} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_{1\text{PN}}}{x_{1\text{PN}}} \right) + \frac{1}{2} \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_{1\text{PN}}}{(x_{1\text{PN}})^3} \right], \end{aligned} \quad (\text{J3})$$

where the new impact vector

$$\hat{\mathbf{d}}_\sigma = \boldsymbol{\sigma} \times (\mathbf{x}_{1\text{PN}} \times \boldsymbol{\sigma}) \quad (\text{J4})$$

has been introduced; note that $\boldsymbol{\sigma} \cdot \hat{\mathbf{d}}_\sigma = 0$ and its absolute value $\hat{d}_\sigma = |\hat{\mathbf{d}}_\sigma|$. In view of Eq. (64) the new impact vector (J4) is determined by

$$\hat{\mathbf{d}}_\sigma = \mathbf{d}_\sigma + \boldsymbol{\sigma} \times (\Delta \mathbf{x}_{1\text{PN}} \times \boldsymbol{\sigma}) \quad (\text{J5})$$

where $\Delta \mathbf{x}_{1\text{PN}}$ is given by Eq.(65) with (66) and (67). For the absolute value of (J5) and the inverse powers one obtains in 1PN approximation

$$\hat{d}_\sigma = d_\sigma + \frac{\mathbf{d}_\sigma \cdot \Delta \mathbf{x}_{1\text{PN}}}{d_\sigma}, \quad (\text{J6})$$

$$\frac{1}{(\hat{d}_\sigma)^n} = \frac{1}{(d_\sigma)^n} - \frac{n}{(d_\sigma)^n} \frac{\mathbf{d}_\sigma \cdot \Delta \mathbf{x}_{1\text{PN}}}{(d_\sigma)^2}. \quad (\text{J7})$$

The relation (J7) is actually the first term of a series expansion where the higher order terms are beyond 1PN approximation. This series expansion has a convergence radius determined by the condition $n |\mathbf{d}_\sigma \cdot \Delta \mathbf{x}_{1PN}| \leq (d_\sigma)^2$. Inserting the monopole term one finds $4 n m x_1 < (d_\sigma)^2$; note that the inverse powers $2 \leq n \leq 6$ according to Eq. (J3). Using the extreme case of Jupiter or even the Sun one finds that this condition is well satisfied for any observer located in the Solar System, and is spoiled for observers if they would have distances of more than several hundred au from the massive Solar System body.

In order to determine the expression $\Delta \mathbf{n}_{2PN}$ in Eq. (124),

$$\Delta \mathbf{n}_{2PN}(\mathbf{x}_N) = \mathbf{n}_{1PN}(\mathbf{x}_N) - \mathbf{n}_{1PN}(\mathbf{x}_{1PN}) + \mathcal{O}(c^{-6}), \quad (\text{J8})$$

where $\mathbf{n}_{1PN}(\mathbf{x}_N)$ and $\mathbf{n}_{1PN}(\mathbf{x}_{1PN})$ are given by Eqs. (J1) and (J3), respectively, one has to perform a series expansion of $\mathbf{n}_{1PN}(\mathbf{x}_{1PN})$ up to terms of the order $\mathcal{O}(c^{-6})$ to be consistent with the 2PN approximation. In this way, by means of relations (E12) - (E14) as well as (J5) - (J7), one obtains

$$\Delta \mathbf{n}_{2PN} = \Delta \mathbf{n}_{2PN}^{M \times M} + \Delta \mathbf{n}_{2PN}^{M \times Q} + \Delta \mathbf{n}_{2PN}^{Q \times Q}, \quad (\text{J9})$$

with

$$\Delta \mathbf{n}_{2PN}^{M \times M}(\mathbf{x}_N) = 4 \frac{G^2 M^2}{c^4} \frac{1}{(d_\sigma)^2} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} \right) \frac{x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N - x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}{d_\sigma}, \quad (\text{J10})$$

and

$$\begin{aligned} \Delta \mathbf{n}_{2PN}^{M \times Q}(\mathbf{x}_N) = & 16 \frac{G^2 M^2}{c^4} J_2 \frac{P^2}{(d_\sigma)^4} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} \right) \frac{x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N - x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}{d_\sigma} \\ & \times \left[\left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 - 3 \left(\frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \right)^2 \right) \frac{\mathbf{d}_\sigma}{d_\sigma} + \frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \mathbf{e}_3 \right], \end{aligned} \quad (\text{J11})$$

and

$$\Delta \mathbf{n}_{2PN}^{Q \times Q}(\mathbf{x}_N) = 12 \frac{G^2 M^2}{c^4} (J_2)^2 \frac{P^4}{(d_\sigma)^6} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} \right) \frac{x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N - x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0}{d_\sigma} \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right)^2 \frac{\mathbf{d}_\sigma}{d_\sigma}, \quad (\text{J12})$$

where in (J11) the quadrupole tensor (F3) and in (J12) the product of two quadrupole tensors (I3) have been inserted. In (J10) - (J12) all terms which are proportional to $\frac{d_\sigma}{x_0}$ and $\frac{d_\sigma}{x_N}$ have been omitted, because they contribute less than 1 nas even for grazing light rays at any massive Solar System body and can be neglected even for high-precision astrometry on the sub- μas and nas-level of accuracy. The calculation of (J9) has revealed that the *enhanced 2PN terms* originate from relations (J5) - (J7), that means they are solely caused by the new impact vector $\hat{\mathbf{d}}_\sigma$ in (J4).

In (J10) - (J12) one can replace the unperturbed position of the light signal at time of reception, $\mathbf{x}_N(t_1)$, by the exact position of the observer, $\mathbf{x}_1$, in view of (122), that means such a replacement in (J10) - (J12) would cause an error of the order $\mathcal{O}(c^{-6})$ which is in line with the 2PN approximation. In this way one finally arrives at the expression as given by Eqs. (133) - (135). By evaluating the absolute value of the light deflection $|\boldsymbol{\sigma} \times \mathbf{n}|$ one finds the upper limits of these *enhanced terms* as given by Eqs. (144) - (146).

Appendix K: Proof of relations (141) - (143)

Finally we will proof the upper limits given in relations (141) - (143). The expression for $\mathbf{n}_{2PN}$ is formally given by Eq. (121), and the light deflection of this term reads

$$\boldsymbol{\sigma} \times \mathbf{n}_{2PN} = \boldsymbol{\sigma} \times \frac{\Delta \dot{\mathbf{x}}_{2PN}}{c} - \left(\frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1PN}}{c} \right) \boldsymbol{\sigma} \times \frac{\Delta \dot{\mathbf{x}}_{1PN}}{c}. \quad (\text{K1})$$

By inspection of Eqs. (81) and (82) one obtains three terms

$$\boldsymbol{\sigma} \times \mathbf{n}_{2PN} = \boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{M \times M} + \boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{M \times Q} + \boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{Q \times Q}, \quad (\text{K2})$$

with

$$\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{M \times M} = \boldsymbol{\sigma} \times \frac{\Delta \dot{\mathbf{x}}_{2PN}^{M \times M}}{c} - \left(\frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1PN}^M}{c} \right) \boldsymbol{\sigma} \times \frac{\Delta \dot{\mathbf{x}}_{1PN}^M}{c}, \quad (K3)$$

$$\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{M \times Q} = \boldsymbol{\sigma} \times \frac{\Delta \dot{\mathbf{x}}_{2PN}^{M \times Q}}{c} - 2 \left(\frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1PN}^Q}{c} \right) \boldsymbol{\sigma} \times \frac{\Delta \dot{\mathbf{x}}_{1PN}^M}{c}, \quad (K4)$$

$$\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{Q \times Q} = \boldsymbol{\sigma} \times \frac{\Delta \dot{\mathbf{x}}_{2PN}^{Q \times Q}}{c} - \left(\frac{\boldsymbol{\sigma} \cdot \Delta \dot{\mathbf{x}}_{1PN}^Q}{c} \right) \boldsymbol{\sigma} \times \frac{\Delta \dot{\mathbf{x}}_{1PN}^Q}{c}, \quad (K5)$$

where the argument $\mathbf{x}_N(t_1)$ can be replaced by the exact spatial position of the observer $\mathbf{x}_1$ in view of (122). The absolute value of (K2) can be estimated by

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2PN}| \leq |\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{M \times M}| + |\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{M \times Q}| + |\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{Q \times Q}|. \quad (K6)$$

By inserting Eqs. (62) - (63) and Eqs. (83) - (85) into (K3) - (K5) one might determine these upper limits in (K6). Such an ambitious calculation is, however, not necessary, because of the following relations:

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{M \times M}| \leq |\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2PN}^{M \times M}| \leq \frac{G^2 M^2}{c^4} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} - 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|, \quad (K7)$$

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{M \times Q}| \leq |\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2PN}^{M \times Q}| \leq 4 \frac{G^2 M^2}{c^4} |J_2| \frac{P^2}{(d_\sigma)^2} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} + 8 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|, \quad (K8)$$

$$|\boldsymbol{\sigma} \times \mathbf{n}_{2PN}^{Q \times Q}| \leq |\boldsymbol{\sigma} \times \boldsymbol{\nu}_{2PN}^{Q \times Q}| \leq 2 \frac{G^2 M^2}{c^4} |J_2|^2 \frac{P^4}{(d_\sigma)^4} \left| \frac{15}{4} \frac{\pi}{(d_\sigma)^2} + 12 \frac{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}{(d_\sigma)^3} \right|, \quad (K9)$$

where we have used the upper limits of the 2PN terms in the total light deflection as given by relations (108) - (110). In view of relation (136), these expressions coincide with the asserted relations in (141) - (143).

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- [91] In celestial mechanics another notation is used (in celestial mechanics the term *post-Newtonian correction* includes $h_{\alpha\beta}^{(2)}$ and $h_{\alpha\beta}^{(3)}$ and $h_{00}^{(4)}$) and, therefore, our notation for the terms in the expansion (4) is sometimes called *light-ray metric*. In our investigation we will call (4) just metric and will drop the term *light-ray metric*.
- [92] For matter we adopt the post-Newtonian statements $T^{\alpha\beta} = \mathcal{O}(c^2, c^1, c^0)$ [76]. The given order in Eqs. (17) - (18) follows from Eqs. (6.2) - (6.3) in [70]; cf. Eqs. (5.11a) and (5.11b) in [77] or Eqs. (99a) - (99b) in [71].
- [93] In the literature the mass-quadrupole moment is sometimes introduced in the following form (e.g. Eq. (3) in [29]) $M_{ab} = \int d^3x x^{ab} (T^{00} + T^{kk})/c^2 + \mathcal{O}(c^{-5})$ where $x^{ab} = x^a x^b$ so that M_{ab} is not traceless: $M_{kk} = \delta^{ab} M_{ab} \neq 0$. However, the trace of the quadrupole moment (or any higher moments) does not contribute to the metric tensor due to STF relation (43). In general, trace terms of mass-moments or spin-moments cancel each other in the metric tensor and, therefore, the calculations are considerably simplified if the metric tensor is expressed in terms of STF multipoles.
- [94] Let us note that the trace terms of the quadrupole moment are not zero but they cancel each other in Eq. (26) in [29], while in (49) there are no trace terms at all.
- [95] By means of $M_{ab} = \hat{M}_{ab} + \delta_{ab} M_{ii}/3$, one may recognize that the trace terms of the quadrupole moment cancel each other in the coefficients given by Eqs. (29) in [29].
- [96] We note that $\ln \frac{x_N - \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0} = -\ln \frac{x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0}$.
- [97] In order to demonstrate that agreement, one has to perform a series expansion of $\mathbf{x}_{1PN}$ which appears as argument in the vectorial functions in [19, 79] (cf. Eqs. (E12) - (E14) in Appendix E).
- [98] If $\boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq 0$ then $x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0 \leq \frac{(d_\sigma)^2}{x_0} \leq d_\sigma$ which follows from (H7) with the minus-sign; note that $1 - \sqrt{1 - a^2} \leq a^2$ for $0 \leq a \leq 1$. If $\boldsymbol{\sigma} \cdot \mathbf{x}_0 > 0$ then $x_0 \leq x_1$ which follows from geometrical considerations (light signal must reach the observer's position). These facts allow us to show (147).
- [99] The terms on the left-hand side of relations (H9) and (H10) consist of scalar products of three-vectors, hence they are independent of the orientation of spatial coordinate axes. So, despite that we have previously assumed that $\mathbf{e}_3$ is parallel to the z-direction (cf. Eqs. (F3) and (I3)), one may rotate these axes such that $\boldsymbol{\sigma}$ is aligned parallel to the x-direction and $\mathbf{d}_\sigma$ to the y-direction, while $\mathbf{e}_3 = (e_3^x, e_3^y, e_3^z)$ has three spatial components now. Then, $\boldsymbol{\sigma} \cdot \mathbf{e}_3 = e_3^x$ and $\frac{\mathbf{d}_\sigma}{d_\sigma} \cdot \mathbf{e}_3 = e_3^y$. Using these relations and the fact that the possible values for the components of $\mathbf{e}_3$ are restricted because $\mathbf{e}_3$ is a unit vector, one may show the validity of the inequalities (H9) and (H10).