

TOTAL LIGHT DEFLECTION IN THE GRAVITATIONAL FIELD OF SOLAR SYSTEM BODIES

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ABSTRACT. The total light deflection represents a concept, which allows one to decide which multipoles need to be implemented in the light trajectory for a given astrometric accuracy. The fundamental quantity of total light deflection is the tangent vector of the light trajectory at future infinity. It has been found that this tangent vector is naturally given by Chebyshev polynomials. It is just this remarkable fact, which allows one to determine strict upper limits of total light deflection for each individual multipole of solar system bodies. Special care is taken about the gauge terms. It is found that these gauge terms vanish at spatial infinity. The results are applied to the case of light deflection in the gravitational fields of Jupiter and Saturn.

1. INTRODUCTION

Angular measurements of stellar objects have made impressive advancements during recent decades. In particular, the astrometry missions *Hipparcos* and *Gaia* of European Space Agency (ESA) have reached the milli-arcsecond (mas) and the micro-arcsecond (μ as) level of accuracy, respectively. The next goal in astrometric science is to arrive at the sub-micro-arcsecond (sub- μ as) or even the nano-arcsecond (nas) scale of accuracy. The objectives of such highly precise measurements are overwhelming, e.g.: detection of earth-like planets, stringent tests of relativity, mapping of dark matter from areas beyond the Milky Way, and direct distance measurements of stellar standard candles up to the closest galaxy clusters; see also (Johnston, 2000).

In fact, several missions have been proposed to ESA, aiming at such levels in astrometric precision, like *Theia* and *Gaia-NIR*, which are primarily designed to study local dark matter properties, to detect Earth-like exoplanets, and to study the physics of highly compact objects (white dwarfs, neutron stars, black holes). A further promising candidate is *NEAT* (Near Infrared Astrometric Telescope), originally designed for an precision of about 50 nas.

The fundamental assignment in relativistic astrometry is the precise interpretation of observational data, which requires an accurate modeling of trajectories of light signals through the curved space-time of the solar system. In view of recent achievements in astrometric angular observations as well as in view of missions proposed to ESA, a corresponding development in the theory of light propagation is indispensable. The investigation of the total light deflection is a further step towards these directions.

2. THE METRIC TENSOR

The curved space-time is described by the pair $(\mathcal{M}, g_{\alpha\beta})$ where $\mathcal{M}$ is a four-dimensional differentiable manifold, while $g_{\alpha\beta}$ is the metric tensor of the manifold, and each point $\mathcal{P} \in \mathcal{M}$ represents a space-time event. The metric tensor is governed by the field equations of gravity (Einstein, 1915), which relate the metric tensor $g_{\alpha\beta}$ of the physical manifold $\mathcal{M}$ to the stress-energy tensor of matter $T_{\alpha\beta}$. These exact field equations can only be solved in closed form for highly symmetric bodies, like spherically symmetric bodies or bodies of ellipsoidal shape, but not for realistic bodies of the solar system. Therefore, approximative approaches of general relativity are essential for further progress in the theory of gravity and in the theory of light

propagation. In the solar system the gravitational fields are weak and, therefore, one may apply the theory of linearized gravity. In that approximation, the covariant components of the metric tensor are decomposed into the flat Minkowski metric $\eta_{\alpha\beta} = (-1, +1, +1, +1)$ plus a metric perturbation $h_{\alpha\beta}$,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad \Longrightarrow \quad \bar{g}^{\alpha\beta} = \eta^{\alpha\beta} - \bar{h}^{\alpha\beta}, \quad (1)$$

where $\bar{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}$ are the contravariant components of the metric density, with $g = \det(g_{\mu\nu})$ being the determinant of the metric. The decomposition (1) implies that the metric perturbations $h_{\alpha\beta}$ can be thought of as symmetric tensorial fields which propagate in the flat background manifold $\mathcal{M}_0$. The metric of the flat background manifold is given by $\eta_{\alpha\beta}$. Thus, the flat background space-time is described by the pair $(\mathcal{M}_0, \eta_{\alpha\beta})$, and the diffeomorphism between the physical manifold $\mathcal{M}$ and the flat background manifold $\mathcal{M}_0$ implies a one-to-one correspondence of the points $\mathcal{Q} \in \mathcal{M}_0$ to the points $\mathcal{P} \in \mathcal{M}$.

The metric perturbation $h_{\alpha\beta}$ and the metric density perturbation $\bar{h}_{\alpha\beta}$ are uniquely related to each other: $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2} \bar{h} \eta_{\alpha\beta}$ with $\bar{h} = \bar{h}^{\mu\nu} \eta_{\mu\nu}$. The weak-field condition $|h_{\alpha\beta}| \ll 1$ inherits $|\bar{h}^{\alpha\beta}| \ll 1$. In linearized gravity, the tensor indices are lowered and raised by the flat Minkowskian metric, e.g. $h^{\alpha\beta} = h_{\mu\nu} \eta^{\mu\alpha} \eta^{\nu\beta}$.

Inserting (1) into the field equations of gravity and keeping terms linear in the metric perturbation, yields the field equations of linearized gravity (cf. Eq. (18.5) in (Misner, Thorne, Wheeler, 1973)). They are considerably simplified by the harmonic gauge, which implies that the coordinates $\{x\}$, which cover the flat background manifold $\mathcal{M}_0$, satisfy the equation $\square x^\mu = 0$, where $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$ is the flat d'Alembertian. Then, the linearized field equations of gravity read

$$\square \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}. \quad (2)$$

Imposing Fock-Sommerfeld boundary conditions ensures a unique solution of (2) in the coordinates $\{x\}$. Though, the harmonic gauge, $\square x^\mu = 0$, does not uniquely determine these coordinates, but allows for small deformations (Box 18.2 in (Misner, Thorne, Wheeler, 1973) or Eq. (3.521) in (Kopeikin, Efroimsky & Kaplan, 2012))

$$x_{\text{can}}^\alpha = x^\alpha + \xi^\alpha(x^\beta), \quad (3)$$

if the vector fields ξ^α satisfy $\square \xi^\alpha = 0$. The label of these new coordinates $\{x_{\text{can}}\}$ abbreviates the term "canonical". The transformation (3) implies a transformation of the metric tensor,

$$g_{\alpha\beta}(t, \mathbf{x}) = \frac{\partial x_{\text{can}}^\mu}{\partial x^\alpha} \frac{\partial x_{\text{can}}^\nu}{\partial x^\beta} g_{\mu\nu}^{\text{can}}(t_{\text{can}}, \mathbf{x}_{\text{can}}). \quad (4)$$

By inserting (3) into (4) and performing a series expansion of the metric tensor on the r.h.s. around the old coordinates $\{x\}$, one obtains (with notation $\partial_\alpha f \equiv \partial f / \partial x^\alpha$):

$$g_{\alpha\beta}(t, \mathbf{x}) = g_{\alpha\beta}^{\text{can}}(t, \mathbf{x}) + \partial_\alpha \xi_\beta(t, \mathbf{x}) + \partial_\beta \xi_\alpha(t, \mathbf{x}), \quad (5)$$

up to terms of higher order, i.e. up to non-linear terms. As stated above, by imposing the Fock-Sommerfeld boundary condition, the solution for the metric tensor $g_{\alpha\beta}$ in (5) is unique. This unique solution can be expressed in terms of six Cartesian symmetric and tracefree (STF) multipoles $\{\hat{M}_L, \hat{S}_L, \hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L\}$ (Thorne, 1980); the *hat* over the multipoles indicates STF. The canonical piece $g_{\alpha\beta}^{\text{can}}$ in (5) depends on two multipoles only: mass-multipoles and spin-multipoles $\{\hat{M}_L, \hat{S}_L\}$. Accordingly, the gauge transformation of the metric tensor, as given by Eq. (5), results in the following form for the metric perturbations ((Thorne, 1980) and (Blanchet & Damour, 1986) and (Damour & Iyer, 1991)):

$$h_{\alpha\beta}(t, \mathbf{x}) = h_{\alpha\beta}^{\text{can}} \left[\hat{M}_L, \hat{S}_L \right] + \partial_\alpha \xi_\beta \left[\hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L \right] + \partial_\beta \xi_\alpha \left[\hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L \right]. \quad (6)$$

The metric of the curved space-time in the exterior of the massive body is assumed to be time-independent. Then, the canonical metric perturbations in (6) are separated into two pieces, $h_{\alpha\beta}^{\text{can}} = h_{\alpha\beta}^{(2)\text{can}} + h_{\alpha\beta}^{(3)\text{can}}$, which are given by

$$h_{00}^{(2)\text{can}} = \frac{2G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{M}_L \hat{\partial}_L \frac{1}{r} \quad \text{and} \quad h_{0i}^{(3)\text{can}} = \frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l l}{(l+1)!} \epsilon_{iab} \hat{S}_{bL-1} \hat{\partial}_{aL-1} \frac{1}{r}, \quad (7)$$

while $h_{ij}^{(2)\text{can}} = h_{00}^{(2)\text{can}} \delta_{ij}$ and the multipoles $\hat{M}_L$ and $\hat{S}_L$ are given by Eqs. (5.33) and (5.35) in (Damour & Iyer, 1991). The gauge functions in (6) have been determined by (Thorne, 1980) and (Blanchet & Damour, 1986) and (Damour & Iyer, 1991) and read:

$$\xi^0 = + \frac{4G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{r} \quad (8)$$

$$\xi^i = - \frac{4G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{iL} \frac{\hat{X}_L}{r} - \frac{4G}{c^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{L-1} \frac{\hat{Y}_{iL-1}}{r} - \frac{4G}{c^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \frac{l}{l+1} \epsilon_{iab} \partial_{aL-1} \frac{\hat{Z}_{bL-1}}{r}. \quad (9)$$

Here, $r = |\mathbf{x}|$, and

$$\hat{\partial}_L = \text{STF}_{i_1 \dots i_l} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_l}}, \quad (10)$$

where the *hat* in $\hat{\partial}_L$ indicates STF operation with respect to the indices $L = i_1 \dots i_l$. The multipoles $\hat{W}_L, \hat{X}_L, \hat{Y}_L, \hat{Z}_L$ of the gauge functions in (8) and (9) are given in (Damour & Iyer, 1991), but their explicit form is not relevant here, because we will show that the gauge terms in (6) have no impact on the unit tangent vector at future infinity and, therefore, no impact on the total light deflection. This result is an example of the general fact that $g_{\alpha\beta}$ and $g_{\alpha\beta}^{\text{can}}$ in (5) are physically equivalent, because they lead to same observables.

3. THE GEODESIC EQUATION

The light signal is assumed to propagate in the flat background manifold $\mathcal{M}_0$ which is covered by harmonic coordinates, $x^\mu = (x^0, x^1, x^2, x^3)$, where the origin of the spatial axes is located at the center of mass of the body. The exact light trajectory can be written in the following form,

$$\mathbf{x}(t) = \mathbf{x}_0 + c(t - t_0) \boldsymbol{\sigma} + \Delta \mathbf{x}(t), \quad (11)$$

where $\Delta \mathbf{x}$ denotes the corrections to the unperturbed light trajectory, $\mathbf{x}_N(t) = \mathbf{x}_0 + c(t - t_0) \boldsymbol{\sigma}$, and N stands for Newtonian (e.g. Kopeikin, Efroimsky & Kaplan, 2012). Furthermore, we introduce the unit tangent vectors along the light trajectory at past and future infinity,

$$\boldsymbol{\sigma} = \frac{\dot{\mathbf{x}}(t)}{c} \Big|_{t \rightarrow -\infty} \quad \text{and} \quad \boldsymbol{\nu} = \frac{\dot{\mathbf{x}}(t)}{c} \Big|_{t \rightarrow +\infty}, \quad (12)$$

where a dot means total derivative with respect to coordinate time, and from (12) follows $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$ and $\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1$. The total light deflection is the angle between these unit vectors,

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}) = \arcsin |\boldsymbol{\sigma} \times \boldsymbol{\nu}|. \quad (13)$$

The evaluation of this quantity is essential, in order to decide which multipoles need to be implemented in the relativistic model of light propagation for a given astrometric accuracy.

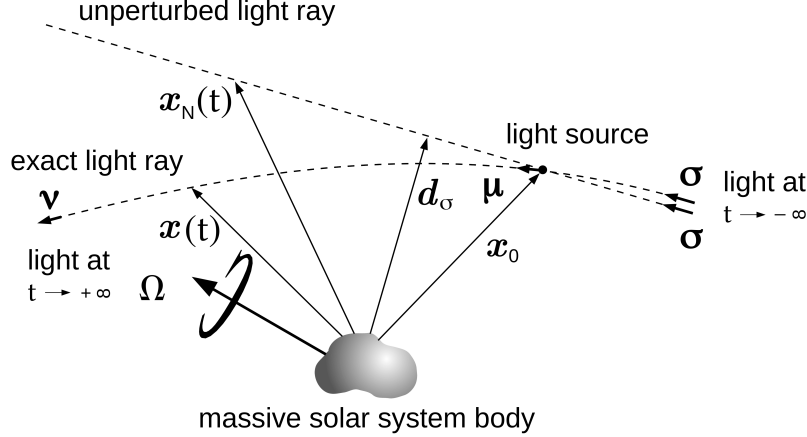


Figure 1: The light signal is emitted by the celestial light source at $\mathbf{x}_0$ in the direction of unit-vector $\boldsymbol{\mu}$ and propagates along the exact trajectory $\mathbf{x}(t)$. The origin of the spatial coordinates is located at the center of mass of the body, and the spatial coordinate axes are aligned with the principal axes of the body. The body is in rotational motion around some axis with angular velocity Ω . The unit tangent vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\nu}$ of the light trajectory at past infinity and future infinity are defined by Eqs. (12), while $\mathbf{d}_\sigma = \boldsymbol{\sigma} \times (\mathbf{x}_0 \times \boldsymbol{\sigma})$ is the impact vector of the unperturbed light ray.

The geodesic equation for light rays in the post-Newtonian (PN) scheme in 1.5PN approximation reads (Kopeikin, Efroimsky & Kaplan, 2012) (with notation $f_{,i} \equiv \partial f / \partial x^i$):

$$\frac{\ddot{x}^i(t)}{c^2} = \frac{1}{2} h_{00,i} - h_{00,j} \sigma^i \sigma^j - h_{ij,k} \sigma^j \sigma^k + \frac{1}{2} h_{jk,i} \sigma^j \sigma^k - h_{0i,j} \sigma^j + h_{0j,i} \sigma^j - h_{0j,k} \sigma^i \sigma^j \sigma^k, \quad (14)$$

where the double-dot means twice the total derivative with respect to the coordinate time. Eq. (14) is valid up to terms of the post-post-Newtonian order $\mathcal{O}(c^{-4})$, and all those terms have been omitted which contain a derivative of the metric perturbations with respect to time, because we consider the stationary case, that is the case of time-independent metric. Note, that in stationary case the geodesic equation in 1.5PN approximation in (14) and the geodesic equation in 1PM approximation of the post-Minkowskian (PM) scheme agree with each other; cf. Eqs. (A.4) and (A.6) in (Klioner & Peip, 2003). If one inserts the metric perturbation (6) into the geodesic equation (14), one may separate the geodesic equations into a canonical term, $\ddot{\mathbf{x}}_{\text{can}}$, plus a gauge term, $\ddot{\mathbf{x}}_{\text{gauge}}$, as follows:

$$\frac{\ddot{\mathbf{x}}(t)}{c^2} = \frac{\ddot{\mathbf{x}}_{\text{can}}(t)}{c^2} + \frac{\ddot{\mathbf{x}}_{\text{gauge}}(t)}{c^2}, \quad (15)$$

where the spatial components of these terms are

$$\frac{\ddot{x}_{\text{can}}^i(t)}{c^2} = h_{00,i}^{(2)\text{can}} - 2 h_{00,j}^{(2)\text{can}} \sigma^i \sigma^j - h_{0i,j}^{(3)\text{can}} \sigma^j + h_{0j,i}^{(3)\text{can}} \sigma^j - h_{0j,k}^{(3)\text{can}} \sigma^i \sigma^j \sigma^k, \quad (16)$$

$$\frac{\ddot{x}_{\text{gauge}}^i(t)}{c^2} = \partial_j \xi_{,k}^0 \sigma^i \sigma^j \sigma^k - \partial_j \xi_{,k}^i \sigma^j \sigma^k. \quad (17)$$

The metric perturbations in (16) are given by (7), while the gauge functions in (17) are given by (8) and (9); notice $\mathbf{x} = \mathbf{x}_N + \mathcal{O}(c^{-2})$ and $r = |\mathbf{x}_N| + \mathcal{O}(c^{-2})$ according to Eq. (11). The first integration of (15) yields the coordinate velocity of the light signal,

$$\frac{\dot{\mathbf{x}}(t)}{c} = \boldsymbol{\sigma} + \frac{\dot{\mathbf{x}}_{\text{can}}(t)}{c} + \frac{\dot{\mathbf{x}}_{\text{gauge}}(t)}{c}, \quad (18)$$

and the unit tangent vectors (12) are obtained from (18) by taking the limit at plus and minus infinity. In the Appendix it is shown that the gauge terms (17) do not contribute to these unit tangent vectors at infinity, because their first time derivative vanishes at plus and minus infinity,

$$\lim_{t \rightarrow \pm\infty} \frac{\dot{\mathbf{x}}_{\text{gauge}}(t)}{c} = 0. \quad (19)$$

Accordingly, only the canonical terms in (16) contribute to the unit tangent vector and, therefore, contribute to the total light deflection.

4. TOTAL LIGHT DEFLECTION IN FIELD OF ARBITRARY BODY

As stated above, the gauge terms in (17) do not contribute to the unit tangent vectors at plus and minus infinity (see Appendix), and there is no need to account for these terms. The first integration of the canonical terms (16) in the geodesic equation has been performed in (Kopeikin, 1997). Taking the limit at plus infinity one arrives at the following expression for the unit tangent vector in (12),

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \boldsymbol{\nu}_{1\text{PN}}^{M_L} + \sum_{l=1}^{\infty} \boldsymbol{\nu}_{1.5\text{PN}}^{S_L} + \mathcal{O}(c^{-4}). \quad (20)$$

The individual terms in (20) are given by (limits of Eqs. (34) and (37) in (Kopeikin, 1997)),

$$\nu_{1\text{PN}}^{i M_L} = -\frac{4G}{c^2} \frac{(-1)^l}{l!} \hat{M}_L P^{ij} \frac{\partial}{\partial \xi^j} \hat{\partial}_L \ln |\boldsymbol{\xi}|, \quad (21)$$

$$\nu_{1.5\text{PN}}^{i S_L} = -\frac{8G}{c^3} \frac{(-1)^l}{l!} \frac{l}{l+1} \sigma^c \epsilon_{i b c} \hat{S}_{b L-1} P^{ij} \frac{\partial}{\partial \xi^j} \hat{\partial}_L \ln |\boldsymbol{\xi}|, \quad (22)$$

where $P^{ij} = \delta^{ij} - \sigma^i \sigma^j$, and $\xi^i = P_j^i x_N^j$ which will later be identified with the impact vector $\mathbf{d}_\sigma$ (cf. text below Eq. (32)). The differential operator in (21) and (22) is given by (cf. Eq. (24) in (Kopeikin, 1997) or Eq. (30) in (Zschocke, 2022))

$$\hat{\partial}_L = \text{STF}_{i_1 \dots i_l} \sum_{p=0}^l \frac{l!}{(l-p)! p!} \sigma_{i_1} \dots \sigma_{i_p} P_{i_{p+1}}^{j_{p+1}} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \left(\frac{\partial}{\partial c\tau} \right)^p. \quad (23)$$

The operator (10) is w.r.t. spatial coordinates x^a , while the operator (23) is w.r.t. new variables $c\tau$ and ξ^a , and the notation *hat* in (10) and *wide hat* in (23) refers to this fact.

Because $\ln |\boldsymbol{\xi}|$ in (21) and (22) is independent of variable $c\tau$, only the term $p = 0$ in (23) is relevant, which considerably simplifies the differential operator in (23). A longer algebraic calculation leads finally to the following remarkable result (Zschocke, 2023):

$$\hat{\partial}_L \ln |\boldsymbol{\xi}| = \frac{(-1)^{l+1}}{|\boldsymbol{\xi}|^l} \text{STF}_{i_1 \dots i_l} \sum_{n=0}^{[l/2]} G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \frac{\xi_{i_{2n+1}} \dots \xi_{i_l}}{|\boldsymbol{\xi}|^{l-2n}}, \quad (24)$$

which is valid for any natural number $l \geq 1$. The scalar coefficients in (24) are given by

$$G_n^l = (-1)^n 2^{l-2n-1} \frac{l!}{n!} \frac{(l-n-1)!}{(l-2n)!}. \quad (25)$$

Remarkably, these coefficients coincide with the coefficients of the power series representation of Chebyshev polynomials of first kind T_l in (28) up to a constant factor $(l-1)!$. In other words,

the expression in (24) is the generating function of the coefficients of Chebyshev polynomials of first kind.

5. TOTAL LIGHT DEFLECTION IN FIELD OF AXISYMMETRIC BODY

In order to determine the mass-multipoles $\hat{M}_L$ and spin-multipoles $\hat{S}_L$ in (21) and (22), the solar system bodies are described by a rigid axisymmetric structure and with arbitrary radial-dependent mass-density. Furthermore, the body is assumed to be in uniform rotational motion around its symmetry axis $\mathbf{e}_3$. For such an axisymmetric body the mass-multipoles and spin-multipoles have been calculated in (Zschocke, 2022) and depend on four physical parameters of the body: mass M , equatorial radius P , zonal harmonic coefficients J_l , angular velocity Ω . Then, it has been shown in (Zschocke, 2023) that for such an axisymmetric body the mass-multipole in (21) and spin-multipole terms in (22) are given by Chebyshev polynomials of first kind and second kind,

$$\nu_{1\text{PN}}^i M_L = -\frac{4GM}{c^2} \frac{J_l}{l} \left[1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right]^{[l/2]} P^{ij} \frac{\partial}{\partial \xi^j} \left(\frac{P}{|\boldsymbol{\xi}|}\right)^l T_l(x), \quad (26)$$

$$\nu_{1\text{PN}}^i S_L = -\frac{8GM}{c^3} \Omega P \frac{J_{l-1}}{l+4} \left[1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right]^{[l/2]} P^{ij} \frac{\partial}{\partial \xi^j} \frac{(\boldsymbol{\sigma} \times \mathbf{d}_\sigma) \cdot \mathbf{e}_3}{d_\sigma} \left(\frac{P}{|\boldsymbol{\xi}|}\right)^l U_{l-1}(x), \quad (27)$$

where the power representations of the Chebyshev polynomials read (Arfken & Weber, 1995),

$$T_l(x) = \frac{l}{2} \sum_{n=0}^{[l/2]} \frac{(-1)^n}{n!} \frac{(l-n-1)!}{(l-2n)!} (2x)^{l-2n} \quad \text{and} \quad U_l(x) = \sum_{n=0}^{[l/2]} \frac{(-1)^n}{n!} \frac{(l-n)!}{(l-2n)!} (2x)^{l-2n}, \quad (28)$$

with $T_0 = 1$. The real variable x in (26) and (27) is defined by

$$x = \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right)^{-1/2} \left(\frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma}\right) \quad \text{where} \quad -1 \leq x \leq +1. \quad (29)$$

It is just this highly remarkable fact, that the tangent vector $\boldsymbol{\nu}$ is given by Chebyshev polynomials, which allows for a strict determination of the upper limits of the angle of total light deflection in (13). This is because the upper limits of Chebyshev polynomials are given by

$$|T_l| \leq 1 \quad \text{and} \quad |U_{l-1}| \leq l. \quad (30)$$

Accordingly, in the 1.5PN approximation the total light deflection (13) is given by

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}) = \sum_{l=0}^{\infty} \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L}) + \sum_{l=1}^{\infty} \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L}). \quad (31)$$

The individual terms are given by ((Kopeikin, 1997), (Klioner, 1991), (Zschocke, 2023))

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L}) = -\boldsymbol{\nu}_{1\text{PN}}^{M_L} \cdot \frac{\mathbf{d}_\sigma}{d_\sigma} \quad \text{and} \quad \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L}) = -\boldsymbol{\nu}_{1.5\text{PN}}^{S_L} \cdot \frac{\mathbf{d}_\sigma}{d_\sigma}, \quad (32)$$

where $\mathbf{d}_\sigma = \boldsymbol{\sigma} \times (\mathbf{x}_0 \times \boldsymbol{\sigma})$ is the impact vector, pointing from the body towards the unperturbed light ray at their closest distance. The absolute value, $d_\sigma = |\mathbf{d}_\sigma|$, is the impact parameter. By inserting (26) and (27) into (32) one obtains the following expressions for the individual mass-multipole and spin-multipole terms in the angle of total light deflection (31) (Zschocke, 2023):

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L}) = -\frac{4GM}{c^2 d_\sigma} J_l \left(\frac{P}{d_\sigma}\right)^l \left[1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right]^{[l/2]} T_l(x), \quad (33)$$

$$\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L}) = -\frac{8GM}{c^3} J_{l-1} \frac{\Omega l}{l+4} \left(\frac{P}{d_\sigma}\right)^{l+1} \frac{(\boldsymbol{\sigma} \times \mathbf{d}_\sigma) \cdot \mathbf{e}_3}{d_\sigma} \left[1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right]^{[l/2]} U_{l-1}(x), \quad (34)$$

where (33) is valid for $l \geq 0$, while (34) is valid for $l \geq 3$. Thus far, it has not been possible to determine the upper limits of the total light deflection terms in (33) and (34), because these scalar functions are pretty much involved. In order to determine their upper limits, one actually would have to calculate their first derivatives with respect to variable x , and then to solve the corresponding algebraic equation of some order n , which is increasing with increasing multipole order l . However, according to the group theory of (Galois, 1846) there exist, in the general case, no radicals for solving such equations for orders $n > 4$. Therefore, it is essential to recognize that the angle of total light deflection is just given in terms of Chebyshev polynomials of first and second kind. Only because of this important fact it is possible to determine the upper limits of (33) and (34) by means of relations (30). Because the impact parameter is larger or equal to the equatorial radius of the body, $d_\sigma \geq P$, one obtains from (33) and (34),

$$\left| \delta(\sigma, \nu_{1\text{PN}}^{M_L}) \right| \leq \frac{4GM}{c^2} \frac{|J_l|}{P} \quad \text{and} \quad \left| \delta(\sigma, \nu_{1.5\text{PN}}^{S_L}) \right| \leq \frac{8GM}{c^3} \Omega \frac{l^2}{l+4} |J_{l-1}|, \quad (35)$$

where the inequality on the l.h.s. and r.h.s. are valid for $l \geq 0$ and $l \geq 3$, respectively; for the case of spin-dipole ($l = 1$) one finds $\left| \delta(\sigma, \nu_{1.5\text{PN}}^{S_1}) \right| \leq \frac{4GM}{c^3} \Omega \kappa^2$ (Klioner, 1991), where κ^2 is the dimensionless moment of inertia. These inequalities (35) for the total light deflection are strictly valid in the 1PN and 1.5PN, and can be used to decide, whether a specific multipole term needs to be taken into account in the light propagation model for a given goal accuracy of future astrometry missions aiming at the sub-micro-arcsecond and nano-arcsecond level. Some numerical values are presented in Table 1 for the case of light deflection of the giant planets Jupiter and Saturn.

Table 1: The upper limits of total light deflection at giant planets Jupiter and Saturn caused by their mass-multipoles and spin-multipoles according to Eqs. (35). All values are given in micro-arcsecond (μas). A blank entry indicates the light deflection is smaller than a nano-arcsecond (nas). For the physical parameters M, P, J_l, Ω standard values are used (Zschocke, 2023).

Light deflection	Jupiter	Saturn	Light deflection	Jupiter	Saturn
$ \delta(\sigma, \nu_{1\text{PN}}^{M_0}) $	16.3×10^3	5.8×10^3	$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_1}) $	0.17	0.04
$ \delta(\sigma, \nu_{1\text{PN}}^{M_2}) $	239	94	$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_3}) $	0.026	0.008
$ \delta(\sigma, \nu_{1\text{PN}}^{M_4}) $	9.6	5.41	$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_5}) $	0.001	—
$ \delta(\sigma, \nu_{1\text{PN}}^{M_6}) $	0.55	0.50	$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_7}) $	—	—
$ \delta(\sigma, \nu_{1\text{PN}}^{M_8}) $	0.04	0.06	$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_9}) $	—	—
$ \delta(\sigma, \nu_{1\text{PN}}^{M_{10}}) $	0.003	0.01	$ \delta(\sigma, \nu_{1.5\text{PN}}^{S_{11}}) $	—	—

6. CONCLUSION

The determination of the upper limits of the angle of total light deflection provides a criterion, up to which order in l the mass-multipoles $\hat{M}_L$ and the spin-multipoles $\hat{S}_L$ need to be taken into account. Such a criterion simplifies considerably the relativistic modeling of light trajectories for future ultra-high precision astrometry missions on the sub- μas level of accuracy. In our investigation we have determined the unit tangent vector of the light ray at future infinity of the light trajectory by Eqs. (26) and (27) as well as strict upper limits for the total light deflection angle by Eqs. (35) for higher mass-multipoles and spin-multipoles. The remarkable fact, that the unit tangent vector of the light ray at future infinity is naturally given by Chebyshev polynomials,

allows for a strict mathematical statement about the upper limits of the total light deflection.

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APPENDIX

In this appendix we will demonstrate the limit (19). We shall use $\mathbf{x} = \mathbf{x}_N + \mathcal{O}(c^{-2})$ and hence $r = |\mathbf{x}_N| + \mathcal{O}(c^{-2})$. The gauge terms in the geodesic equation (17) consist of two pieces, $\ddot{\mathbf{x}}_{\text{gauge}} = \ddot{\mathbf{x}}_{\text{g1}} + \ddot{\mathbf{x}}_{\text{g2}}$. Their spatial components are given by

$$\frac{\ddot{x}_{\text{g1}}^i(t)}{c^2} = +\partial_j \xi_{,k}^0 \sigma^i \sigma^j \sigma^k \quad \text{and} \quad \frac{\ddot{x}_{\text{g2}}^i(t)}{c^2} = -\partial_j \xi_{,k}^i \sigma^j \sigma^k, \quad (36)$$

where the gauge vectors are given by Eqs. (8) and (9). Let us consider the first term in (36). Using $(r^{-1})_{,jk} = 3x_j x_k / r^5 - \delta_{jk} / r^3$, one obtains

$$\frac{\ddot{\mathbf{x}}_{\text{g1}}(t)}{c^2} = +\frac{8G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{r^3} \boldsymbol{\sigma} - \frac{12G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{r^5} (d_\sigma)^2 \boldsymbol{\sigma}, \quad (37)$$

where $(\boldsymbol{\sigma} \cdot \mathbf{x})^2 = r^2 - (d_\sigma)^2 + \mathcal{O}(c^{-2})$ has been used. This expression has to be integrated over the time variable. To apply the advanced integration methods developed by (Kopeikin, 1997), we have to transform (37) from $(ct, \mathbf{x})$ into terms of two new variables, $c\tau = \boldsymbol{\sigma} \cdot \mathbf{x}_N$ and $\xi^i = P^{ij} x_N^j$, which are independent of each other, and obtain (note that $\boldsymbol{\xi} = \mathbf{d}_\sigma$ hence $(d_\sigma)^2 = \boldsymbol{\xi} \cdot \boldsymbol{\xi} = \xi^2$)

$$\frac{\ddot{\mathbf{x}}_{\text{g1}}(\tau)}{c^2} = +\frac{4G}{c^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{W}_L \hat{\partial}_L \left(\frac{2}{(r_N)^3} - \frac{3(\xi)^2}{(r_N)^5} \right) \boldsymbol{\sigma}, \quad (38)$$

where the double-dot in (38) means twice the total derivative with respect to variable τ . The differential operator (38) has been given by Eq. (23). To get the coordinate velocity of the light signal, one has to integrate (38) over variable $c\tau$ (note that $dc\tau = dct$) and obtains for the spatial components

$$\frac{\dot{x}_{\text{g1}}^i}{c} = +\frac{4G}{c^3} \frac{\partial}{\partial c\tau} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{W}_L}{r} \sigma^i. \quad (39)$$

A similar calculation can be performed for the second gauge term in (36), which yields

$$\frac{\dot{x}_{\text{g2}}^i}{c} = -\frac{4G}{c^2} \frac{\partial}{\partial c\tau} \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{iL} \frac{\hat{X}_L}{r} + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_{L-1} \frac{\hat{Y}_{iL}}{r} + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \frac{l}{l+1} \epsilon_{iab} \hat{\partial}_{aL-1} \frac{\hat{Z}_{bL-1}}{r} \right). \quad (40)$$

By inserting (23) into (39) and (40) one finds that these terms vanish at infinity, and we get

$$\lim_{\tau=\pm\infty} \frac{\dot{\mathbf{x}}_{\text{gauge}}(\tau)}{c} = \lim_{\tau=\pm\infty} \frac{\dot{\mathbf{x}}_{\text{g1}}(\tau)}{c} + \lim_{\tau=\pm\infty} \frac{\dot{\mathbf{x}}_{\text{g2}}(\tau)}{c} = 0. \quad (41)$$

Thus, by transforming (41) back from $(c\tau, \boldsymbol{\xi})$ into $(ct, \mathbf{x})$, we have shown the validity of (19).

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