

Total light deflection in the gravitational field of solar system bodies

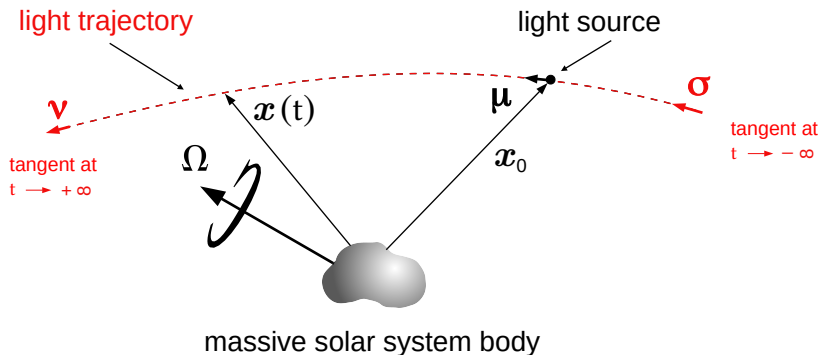
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1. Introduction

a primary task of relativistic astrometry is determination of light trajectories which propagate through the solar system



- angle $\delta(\sigma, \nu) = \arcsin |\sigma \times \nu|$ is the total light deflection

2. Scheme for calculation of light trajectories

weak gravitational fields + canonical harmonic gauge

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}^{(2)} \underbrace{\left[\hat{M}_L \right]}_{\text{shape}} + h_{\alpha\beta}^{(3)} \underbrace{\left[\hat{S}_L \right]}_{\text{rotation}} + \mathcal{O}\left(c^{-4}\right)$$

geodesic equation for light rays

$$\frac{\dot{\boldsymbol{x}}(t)}{c} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \frac{\Delta \dot{\boldsymbol{x}}_{1\text{PN}}^{M_L}}{c} + \sum_{l=1}^{\infty} \frac{\Delta \dot{\boldsymbol{x}}_{1.5\text{PN}}^{S_L}}{c} + \mathcal{O}\left(c^{-4}\right)$$

$$\Rightarrow \quad \boldsymbol{\nu} = \left. \frac{\dot{\boldsymbol{x}}(t)}{c} \right|_{t \rightarrow +\infty}$$

3. Multipole decomposition of tangent vector

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \boldsymbol{\nu}_{1\text{PN}}^{M_L} + \sum_{l=1}^{\infty} \boldsymbol{\nu}_{1.5\text{PN}}^{S_L} + \mathcal{O}(c^{-4})$$

- mass-multipole term of tangent vector [Kopeikin (1997)]

$$\nu_{1\text{PN}}^{i M_L} = -\frac{4G}{c^2} P^{ij} \frac{\partial}{\partial \xi^j} \frac{(-1)^l}{l!} \hat{M}_L \hat{\partial}_L \ln |\boldsymbol{\xi}|$$

- spin-multipole term of tangent vector [Kopeikin (1997)]

$$\nu_{1.5\text{PN}}^{i S_L} = -\frac{8G}{c^3} P^{ij} \frac{\partial}{\partial \xi^j} \sigma^c \epsilon_{i l b c} \frac{(-1)^l}{(l+1)!} \hat{S}_{bL-1} \hat{\partial}_L \ln |\boldsymbol{\xi}|$$

- differential operation is defined by

$$\hat{\partial}_L \ln |\boldsymbol{\xi}| = \text{STF}_{i_1 \dots i_l} P_{i_1}^{j_1} \dots P_{i_l}^{j_l} \frac{\partial}{\partial \xi^{j_1}} \dots \frac{\partial}{\partial \xi^{j_l}} \ln |\boldsymbol{\xi}|$$

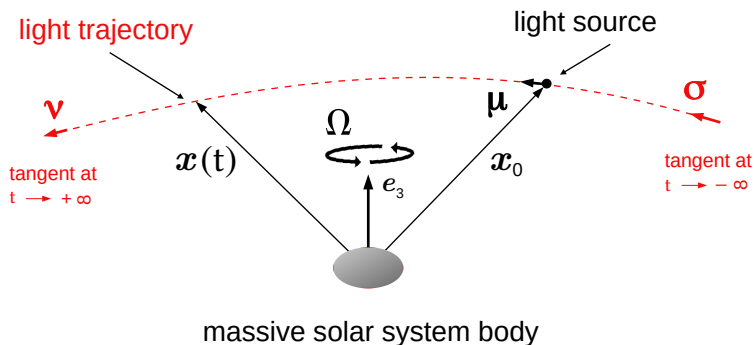
- performing these derivatives yields [Zschocke (2023)]

$$\hat{\partial}_L \ln |\boldsymbol{\xi}| = \frac{(-1)^{l+1}}{|\boldsymbol{\xi}|^l} \text{STF}_{i_1 \dots i_l} \sum_{n=0}^{[l/2]} G_n^l P_{i_1 i_2} \dots P_{i_{2n-1} i_{2n}} \frac{\xi_{i_{2n+1}} \dots \xi_{i_l}}{|\boldsymbol{\xi}|^{l-2n}}$$

- where G_n^l are just coefficients of Chebyshev polynomials

$$G_n^l = (-1)^n 2^{l-2n-1} \frac{l!}{n!} \frac{(l-n-1)!}{(l-2n)!}$$

4. Total light deflection for axisymmetric body



- body in uniform rotation around its symmetry axis e_3
- multipoles $\hat{M}_L$ and $\hat{S}_L$ calculated in their explicit form

- mass-multipole term of tangent vector [Zschocke (2023)]

$$\nu_{1\text{PN}}^{i M_L} = -\frac{4GM}{c^2} \frac{J_l}{l} \left[1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right]^{[l/2]} \times P^{ij} \frac{\partial}{\partial d_\sigma^j} \left(\frac{P}{d_\sigma} \right)^l T_l(x)$$

- $\nu_{1\text{PN}}^{M_L}$ expressed by Chebyshev polynomials of first kind

$$T_l(x) = \frac{l}{2} \sum_{n=0}^{[l/2]} \frac{(-1)^n}{n!} \frac{(l-n-1)!}{(l-2n)!} (2x)^{l-2n}$$

- $P \dots$ equatorial radius , $J_l \dots$ zonal harmonic coefficients
- variable x defined by

$$x = \left(1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2 \right)^{-1/2} \left(\frac{\mathbf{d}_\sigma \cdot \mathbf{e}_3}{d_\sigma} \right)$$

- spin-multipole term of tangent vector [Zschocke (2023)]

$$\nu_{1.5\text{PN}}^{i S_L} = -\frac{8GM}{c^3} \Omega P \frac{J_{l-1}}{l+4} \frac{(\boldsymbol{\sigma} \times \mathbf{d}_\sigma) \cdot \mathbf{e}_3}{d_\sigma} \left[1 - (\boldsymbol{\sigma} \cdot \mathbf{e}_3)^2\right]^{[l/2]} \\ \times P^{ij} \frac{\partial}{\partial d_\sigma^j} \left(\frac{P}{d_\sigma}\right)^l U_{l-1}(x)$$

- $\nu_{1.5\text{PN}}^{S_L}$ expressed by Chebyshev polynomials of second kind

$$U_l(x) = \sum_{n=0}^{[l/2]} \frac{(-1)^n}{n!} \frac{(l-n)!}{(l-2n)!} (2x)^{l-2n}$$

5. The upper limit of total light deflection

The fact that the tangent vector $\boldsymbol{\nu}$ is related to **Chebyshev polynomials** allows to determine the upper limit of angle of total light deflection

- upper limit of total light deflection: mass multipoles

$$|T_l(x)| \leq 1 \quad \implies \quad \left| \delta \left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1PN}}^{M_L} \right) \right| \leq \frac{4GM}{c^2} \frac{|J_l|}{d_\sigma}$$

- upper limit of total light deflection: spin multipoles

$$|U_{l-1}(x)| \leq l \quad \implies \quad \left| \delta \left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1.5PN}}^{S_L} \right) \right| \leq \frac{8GM}{c^3} \Omega \frac{l^2}{l+4} |J_{l-1}|$$

6. Numerical values of total light deflection [μas]

Light deflection	Sun	Jupiter	Saturn
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_0}) $	1.75×10^6	16.3×10^3	5.8×10^3
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_2}) $	0.35	239	94
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_4}) $	1.72	9.6	5.41
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_6}) $	0.07	0.55	0.50
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_8}) $	0.007	0.04	0.06
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_{10}}) $	—	0.003	0.01

Light deflection	Sun	Jupiter	Saturn
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_1}) $	0.7	0.17	0.04
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_3}) $	—	0.026	0.008
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_5}) $	—	0.001	—

7. Summary and Outlook

- tangent vector given in terms of Chebyshev polynomials
- this fact allows to find upper limits of total light deflection
- total light deflection not maximal value of light deflection
- therefore: case of finite distance under investigation now

- [1] K.S. Thorne, *Multipole expansion of gravitational radiation*, Rev. Mod. Phys. **52** (1980) 299.
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- [3] T. Damour, B.R. Iyer, *Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors*, Phys. Rev. D **43** (1991) 3259.
- [4] S.M. Kopeikin, *Propagation of light in the stationary field of multipole gravitational lens*, J. Math. Phys. **38** (1997) 2587.
- [5] S. Zschocke, *Total light deflection in the gravitational field of an axisymmetric body at rest with full mass and spin multipole structure*, Phys. Rev. D **107** (2023) 124055.

8. The metric and tangent vector of light ray

- metric in case of weak gravitational fields

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \quad \text{with} \quad \|h_{\alpha\beta}\| \ll 1$$

- linearized gravity in harmonic gauge

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \left(h_{\alpha\beta} - \frac{1}{2} h^\sigma{}_\sigma \eta_{\alpha\beta} \right) = -\frac{16\pi G}{c^4} T_{\alpha\beta}$$

- post-Newtonian expansion

$$h_{\alpha\beta} = h_{\alpha\beta}^{(2)}{}_{\text{can}} + h_{\alpha\beta}^{(3)}{}_{\text{can}} + \underbrace{\partial_\alpha w_\beta + \partial_\beta w_\alpha}_{\text{gauge-terms}} + \mathcal{O}(c^{-4})$$

- metric perturbations in canonical harmonic gauge
[Thorne (1980), Blanchet, Damour, Iyer (1986, 1991)]

$$\begin{aligned}
 h_{00}^{(2)}{}_{\text{can}} &= \frac{2}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{M}_L}{r} \\
 h_{0i}^{(3)}{}_{\text{can}} &= \frac{4}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l l}{(l+1)!} \epsilon_{iab} \hat{\partial}_{aL-1} \frac{\hat{S}_{bL-1}}{r} \\
 h_{ij}^{(2)}{}_{\text{can}} &= \frac{2}{c^2} \delta_{ij} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{\partial}_L \frac{\hat{M}_L}{r}
 \end{aligned}$$

- mass multipoles: $\hat{M}_L = \int d^3x \hat{x}_L T^{00}/c^2$
- spin multipoles: $\hat{S}_L = \int d^3x \epsilon_{jk< i_l} \hat{x}_{L-1>} x^j T^{0k}/c$

- geodesic equation in 1.5PN approximation

$$\frac{\ddot{x}_i}{c^2} = h_{00,i}^{(2)} - 2 h_{00,j}^{(2)} \sigma_i \sigma^j + h_{0j,i}^{(3)} \sigma^j - h_{0i,j}^{(3)} \sigma^j - h_{0j,k}^{(3)} \sigma_i \sigma^j \sigma^k$$

- advanced integration methods based on new variables

$$c\tau = \boldsymbol{\sigma} \cdot \mathbf{x}_N \quad \text{and} \quad \xi^i = P_j^i x_N^j$$

- solution of geodesic equation in 1.5PN approximation
[Kopeikin (1997)]

$$\begin{aligned} \frac{\dot{\mathbf{x}}}{c} &= \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \frac{\Delta \dot{\mathbf{x}}_{1\text{PN}}^{M_L}}{c} + \sum_{l=1}^{\infty} \frac{\Delta \dot{\mathbf{x}}_{1.5\text{PN}}^{S_L}}{c} \\ \mathbf{x} &= \mathbf{x}_N + \sum_{l=0}^{\infty} \Delta \mathbf{x}_{1\text{PN}}^{M_L} + \sum_{l=1}^{\infty} \Delta \mathbf{x}_{1.5\text{PN}}^{S_L} \end{aligned}$$