

Beyond GREM

Relativity for higher accuracy

Sven Zschocke
Lohrmann-Observatory, TU Dresden, Germany

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1. Introduction

1.1 The new era of space astrometry missions by ESA



Hipparcos
launched: 1989

milli-arcsecond
astrometry (VIS)



Gaia
launched: 2013

micro-arcsecond
astrometry (VIS)



Several space astrometry missions proposed to ESA

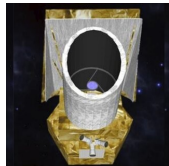
GaiaNIR
ideal case: 2045

micro-arcsecond
astrometry (IR)



Theia
ideal case: 2045

sub-micro-arcsecond
astrometry (VIS)

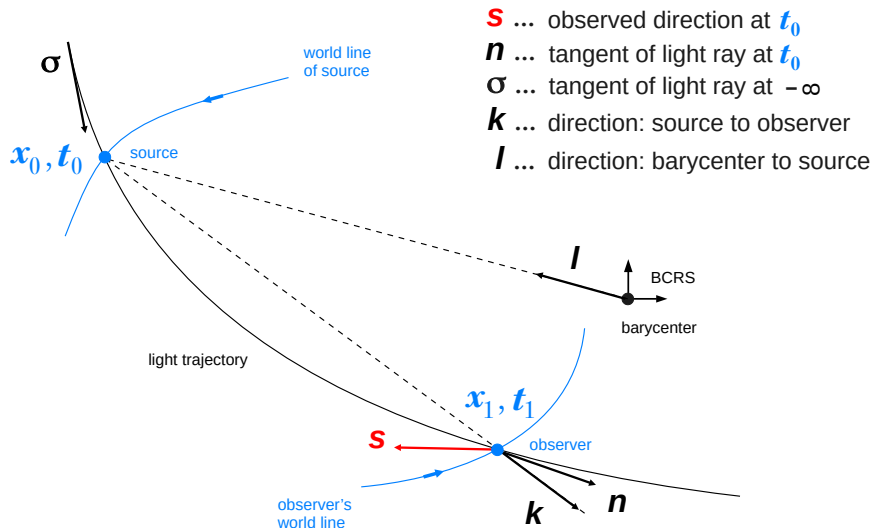


1.2 Science cases of sub-micro-arcsecond astrometry

Science themes of sub- μ as-astrometry are overwhelming

- discovery of Earth-like exoplanets
 - detection of gravitational waves
 - measurements of dark matter distributions
 - new precise test of general relativity
 - extension of model-independent distance ladder
 - and many many more ...
-
- Comment:
theoretical light propagation model should be 10-times
more accurate than end-of-mission astrometric accuracy

1.3 The General Relativistic Model (GREM)



GREM: transformations among these vectors

$$\boldsymbol{s} \rightarrow \boldsymbol{n} \rightarrow \boldsymbol{\sigma} \rightarrow \boldsymbol{k} \rightarrow \boldsymbol{l}$$

- Development of GREM:

S.A. Klioner, AJ **125** (2003) 1580.

S.A. Klioner, PRD **69** (2004) 124001.

- Refinements of GREM:

S.A. Klioner, S. Zschocke, CQG **27** (2010) 075015.

S. Zschocke, S.A. Klioner, CQG **28** (2011) 015009.

Final aim of GREM: position of the source $\boldsymbol{x}_0$

Three important effects

1. **Aberration** described by: $s \rightarrow n$
is related to the motion of observer
2. **Parallax** described by: $k \rightarrow l$
is related to distance of observer and barycenter
3. **Light deflection** described by: $n \rightarrow \sigma \rightarrow k$
they are independent of an observer

Talk is focused on effect of light deflection

Light deflection described by sequence: $\boldsymbol{n} \rightarrow \boldsymbol{\sigma} \rightarrow \boldsymbol{k}$

Tangent vectors determined by light trajectory: $\dot{\boldsymbol{x}}(t), \boldsymbol{x}(t)$

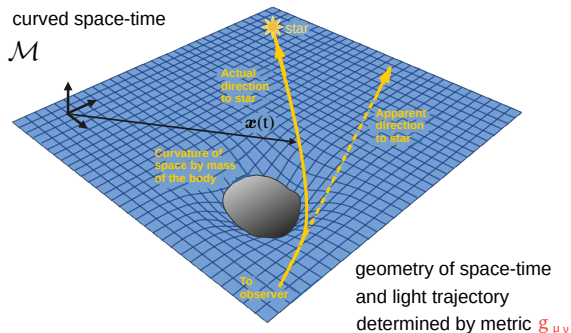
$$\boldsymbol{\sigma} = \lim_{t \rightarrow -\infty} \frac{\dot{\boldsymbol{x}}(t)}{c} \quad (1)$$

$$\boldsymbol{k} = \frac{\boldsymbol{x}(t_1) - \boldsymbol{x}(t_0)}{|\boldsymbol{x}(t_1) - \boldsymbol{x}(t_0)|} \quad (2)$$

$$\boldsymbol{n} = \frac{\dot{\boldsymbol{x}}(t_1)}{|\dot{\boldsymbol{x}}(t_1)|} \quad (3)$$

2. Theory of light propagation

- space-time is curved due to mass and energy of matter

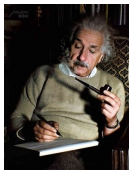


- astrometry needs to determine light trajectories $\mathbf{x}(t)$

- light trajectories are governed by geodesic equation

$$\frac{\ddot{x}^i}{c^2} + \Gamma_{\mu\nu}^i \frac{\dot{x}^\mu}{c} \frac{\dot{x}^\nu}{c} - \Gamma_{\mu\nu}^0 \frac{\dot{x}^\mu}{c} \frac{\dot{x}^\nu}{c} \frac{\dot{x}^i}{c} = 0 \quad \Rightarrow \quad \mathbf{x}(t) \quad (4)$$

- historically: geodesic equation for light (Einstein, 1914)



KÖNIGLICH PREUSSISCHEN
AKADEMIE DER WISSENSCHAFTEN.
Gesamtsitzung vom 19. November.
Mitth. aus der Sitzung der phys.-math. Klasse vom 22. October.

Die formale Grundlage der allgemeinen
Relativitätstheorie.
Von A. EINSTEIN.

welchens Ableitung der euklidischen Gleichungen dieser Linie hier
bestehen.

Es handelt sich um eine Ableitung von Punkten P^0 und P^1 von
einer Linie, gegeben durch die zwei unendlich benachbarten Punkte,
die durch dieselben, gemeinsamen Punkte gehen, die Gleichung (1) ist
dieser. Dasselbe gilt mit γ einer Funktion der Koordinaten x^i ,
und eine "Feld" von Kurven γ auf einer dieser unendlich benach-
barten Punkte P^0 und P^1 bestimmt, deren Koordinaten bei
gegebenen Kurve die Funktionen γ von x^i aufzuheben sind. Setzen
wir

$$x^i = \sum_{\alpha=0}^n x^i_{\alpha} \frac{dx^i}{dx^i}$$

so können wir auch (1) setzen

$$\sum_{\alpha=0}^n x^i_{\alpha} \frac{dx^i}{dx^i} = 1, \quad (10)$$

da die Integrationskurven γ und γ die als bestimmten Kurven be-
zeichnet sind. Beachtet man nun die Punkte, welche man die
K. setzen will, so ist eine Funktion der allgemeinen Gleichungen
Lini in dem von jedem Punkt von γ gebildeten Punkte über der
verbleibenden Linie zu setzen, so hat man

$$dx^i = \left(1 + \sum_{\alpha=0}^n \frac{dx^i_{\alpha}}{dx^i} \frac{dx^i}{dx^i} + \sum_{\alpha=0}^n \frac{dx^i_{\alpha}}{dx^i} \frac{dx^i}{dx^i}\right) dx^i$$

Setzt man dies in (10) ein, so erhält man, indem man die linke
Gleichung partiell integriert und dabei berücksichtigt, daß für $\alpha = 1$, und
 $\alpha = n$, die dx^i_{α} verschwinden

$$\sum_{\alpha=0}^n x^i_{\alpha} \frac{dx^i}{dx^i} = 1,$$

woher

$$K = \sum_{\alpha=0}^n \left(\frac{dx^i_{\alpha}}{dx^i} \frac{dx^i}{dx^i} - \frac{1}{2} \frac{dx^i_{\alpha}}{dx^i} \frac{dx^i}{dx^i} \right)$$

gemäß (9). Es folgt hieraus, daß

$$K = 0$$

die Gleichung des geodetischen Lini ist.

In der ursprünglichen Schreibweise entspricht demselben
geodetischen Lini, die welche $dx^i = 0$ ist, die Bewegung unendlich
benachbarten Punkte, die welche $dx^i = 0$ ist, das Feldlinien. Das

Albert Einstein:

Die Grundlage der allgemeinen Relativitätstheorie

Preussische Akademie der Wissenschaften 2 (1914) 1030.

- Christoffel symbols:

$$\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right) \quad (5)$$

- metric tensor $g_{\mu\nu}$ from field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \quad (6)$$

$R_{\mu\nu}$... Ricci tensor

R ... Ricci scalar

$T_{\mu\nu}$... energy-momentum tensor of matter

$(\sqrt{-g} g^{\mu\nu})_{,\nu} = 0$... harmonic coordinates

Post-Newtonian (PN) approximation scheme

weak-field slow-motion approximation

Post-Newtonian expansion of metric tensor in 1.5PN

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{1\text{PN}} + h_{\mu\nu}^{1.5\text{PN}} + \mathcal{O}(c^{-4}) \quad (7)$$

First integration of geodesic equation in 1.5PN (velocity)

$$\frac{\dot{\boldsymbol{x}}(t)}{c} = \boldsymbol{\sigma} + \frac{\Delta\dot{\boldsymbol{x}}_{1\text{PN}}}{c} + \frac{\Delta\dot{\boldsymbol{x}}_{1.5\text{PN}}}{c} + \mathcal{O}(c^{-4}) \quad (8)$$

Second integration of geodesic equation in 1.5PN (trajectory)

$$\boldsymbol{x}(t) = \boldsymbol{x}_N + \Delta\boldsymbol{x}_{1\text{PN}} + \Delta\boldsymbol{x}_{1.5\text{PN}} + \mathcal{O}(c^{-4}) \quad (9)$$

- Transformation $\mathbf{k} \rightarrow \boldsymbol{\sigma}$ is given by:

$$\boldsymbol{\sigma} = \mathbf{k} - \frac{1}{R} \mathbf{k} \times \left(\frac{\Delta \mathbf{x}_{\text{PN}}(t_1, t_0)}{c} \times \mathbf{k} \right) \quad (10)$$

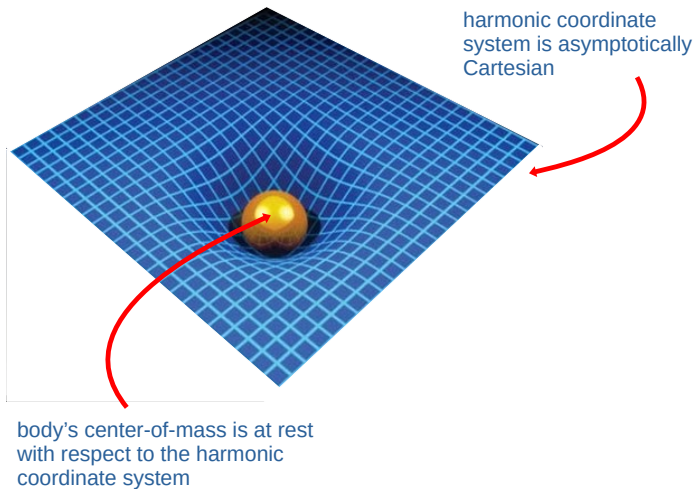
- Transformation $\boldsymbol{\sigma} \rightarrow \mathbf{n}$ is given by:

$$\mathbf{n} = \boldsymbol{\sigma} + \boldsymbol{\sigma} \times \left(\frac{\Delta \dot{\mathbf{x}}_{\text{PN}}(t_1)}{c} \times \boldsymbol{\sigma} \right) \quad (11)$$

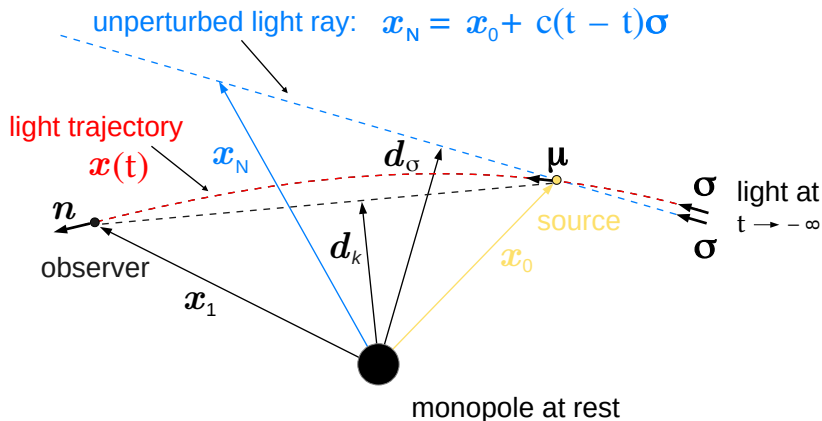
- Transformation $\mathbf{k} \rightarrow \mathbf{n}$ is given by:

$$\mathbf{n} = \mathbf{k} + \mathbf{k} \times \left(\frac{\Delta \dot{\mathbf{x}}_{\text{PN}}}{c} \times \mathbf{k} \right) - \frac{1}{R} \mathbf{k} \times \left(\frac{\Delta \mathbf{x}_{\text{PN}}}{c} \times \mathbf{k} \right) \quad (12)$$

3. Light deflection in the solar system



3.1 Light propagation in field of monopole at rest



- geodesic equation (4) in 1PN for monopole

$$\ddot{\mathbf{x}} = -\frac{2GM}{r^3} \mathbf{x} + 4GM \frac{\boldsymbol{\sigma} \cdot \mathbf{x}}{r^3} \boldsymbol{\sigma} \quad (13)$$

- first integration of geodesic equation (13) in 1PN

$$\frac{\Delta \dot{\mathbf{x}}_{1\text{PN}}^M}{c} = -\frac{2GM}{c^2} \frac{\boldsymbol{\sigma}}{x_N} - \frac{2GM}{c^2} \frac{d_\sigma}{(d_\sigma)^2} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_N} + 1 \right)$$

- second integration of geodesic equation (13) in 1PN

$$\Delta \mathbf{x}_{1\text{PN}}^M = -\frac{2GM}{c^2} \boldsymbol{\sigma} \ln \frac{x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N}{x_0 + \boldsymbol{\sigma} \cdot \mathbf{x}_0} - \frac{2GM}{c^2} \frac{d_\sigma}{(d_\sigma)^2} (x_N + \boldsymbol{\sigma} \cdot \mathbf{x}_N - x_0 - \boldsymbol{\sigma} \cdot \mathbf{x}_0)$$

- vector $\mathbf{n}$ follows from Eq. (12):

$$\mathbf{n}_{1\text{PN}}^M = \mathbf{k} - 2 \frac{GM}{c^2} \frac{\mathbf{d}_k}{(d_k)^2} \frac{x_0 x_1 - \mathbf{x}_0 \cdot \mathbf{x}_1}{x_1 R} \quad (14)$$

- light deflection is the angle between $\mathbf{n}$ and $\mathbf{k}$

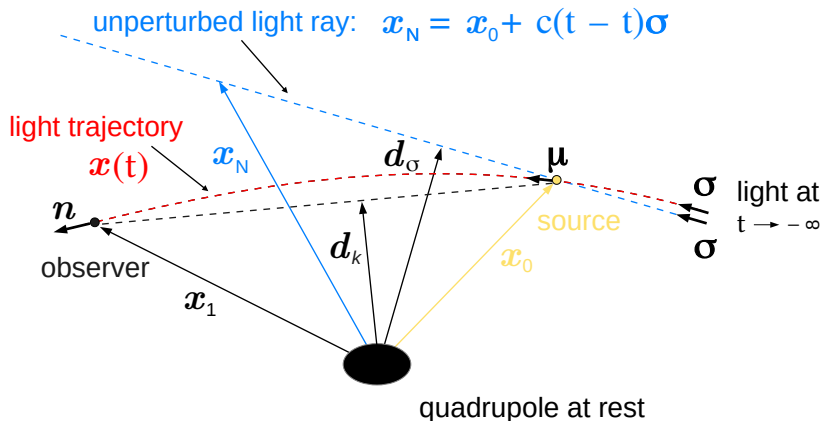
$$\delta(\mathbf{k}, \mathbf{n}_{1\text{PN}}^M) = \arcsin \left| \mathbf{k} \times \mathbf{n}_{1\text{PN}}^M \right| \quad (15)$$

- magnitude of light deflection is criterion for GREM
- for monopole one obtains for the upper limit:

$$\left| \delta(\mathbf{k}, \mathbf{n}_{1\text{PN}}^M) \right| \leq \frac{4GM}{c^2} \frac{1}{d_k} \quad (16)$$

(Sun: 1.75 arcsec predicted by Einstein in 1915)

3.2 Light propagation in field of quadrupole at rest



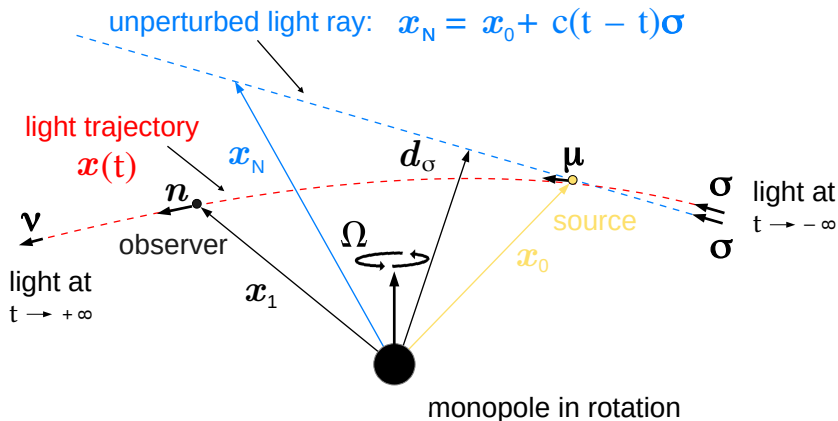
- light trajectory $\Delta \dot{\mathbf{x}}_{\text{IPN}}^Q$ and $\Delta \mathbf{x}_{\text{IPN}}^Q$ determined by
S. Klioner, Sov. Astronomy **35** (1991) 523
these are pretty much involved expressions
- upper limit of light deflection caused by quadrupole:
S. Zschocke, S. Klioner: CQG **28** (2011) 015009

$$\left| \delta \left(\mathbf{k}, \mathbf{n}_{\text{IPN}}^Q \right) \right| \leq \frac{4GM}{c^2} \frac{|J_2|}{d_k} \left(\frac{P}{d_k} \right)^2 \quad (17)$$

P ... equatorial radius of body

J_2 ... second zonal harmonic coefficient of body

3.3 Light propagation in field of spin-dipole at rest



- light trajectory $\Delta \dot{\mathbf{x}}_{\text{IPN}}^S$ and $\Delta \mathbf{x}_{\text{IPN}}^S$ determined by
S. Klioner, Sov. Astronomy **35** (1991) 523
- upper limit of angle $\delta(\mathbf{k}, \mathbf{n}_{1.5\text{PN}}^S)$ has not been calculated
- upper limit of total light deflection caused by spin dipole
S. Klioner, Sov. Astronomy **35** (1991) 523
S. Kopeikin, B. Mashhoon, PRD **65** (2002) 064025
S. Zschocke, PRD **107** (2023) 124055

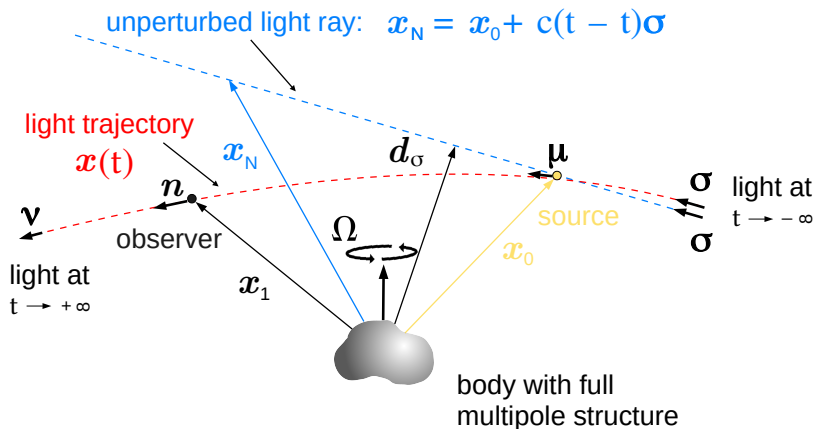
$$\left| \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^S) \right| \leq \frac{4GM}{c^3} \Omega \kappa^2 \quad (18)$$

Ω ... angular velocity of body

κ^2 ... dimensionless moment of inertia

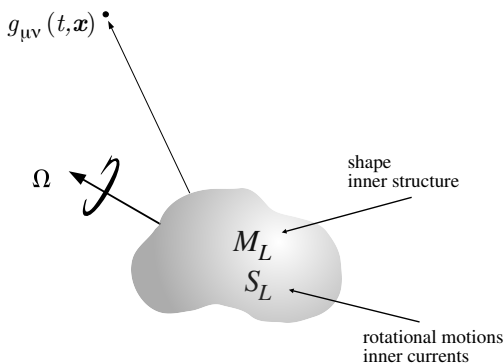
3.4 Light propagation in field of multipoles at rest

- solar system bodies can be of arbitrary shape and structure



Metric of such a body given in terms of multipoles

- $\hat{M}_L$... shape and inner structure
- $\hat{S}_L$... rotation and inner currents



Metric in 1.5PN approximation: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{1\text{PN}} + h_{\mu\nu}^{1.5\text{PN}}$

K. Thorne, Rev.Mod.Phys. **52** (1980) 299

L. Blanchet, T. Damour, Phil.Trans.R.Soc.L. A **320** (1986) 30

T. Damour, B. Iyer, PRD **43** (1991) 3259

$$\begin{aligned}h_{00}^{1\text{PN}}(t, \mathbf{x}) &= \frac{2G}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \frac{\hat{M}_L}{r} \\h_{0i}^{1.5\text{PN}}(t, \mathbf{x}) &= \frac{4G}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l l}{(l+1)!} \epsilon_{iab} \partial_{aL-1} \frac{\hat{S}_{bL-1}}{r} \\h_{ij}^{1\text{PN}}(t, \mathbf{x}) &= \frac{2G}{c^2} \delta_{ij} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \frac{\hat{M}_L}{r}\end{aligned}$$

where $\partial_L = \frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^l}$ and $r = |\mathbf{x}|$

- first integration of geodesic equation (4)

$$\frac{\dot{\boldsymbol{x}}(t)}{c} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \frac{\Delta \dot{\boldsymbol{x}}_{1\text{PN}}^{M_L}}{c} + \sum_{l=1}^{\infty} \frac{\Delta \dot{\boldsymbol{x}}_{1.5\text{PN}}^{S_L}}{c} + \mathcal{O}(c^{-4}) \quad (19)$$

- second integration of geodesic equation (4)

$$\boldsymbol{x}(t) = \boldsymbol{x}_N + \sum_{l=0}^{\infty} \Delta \boldsymbol{x}_{1\text{PN}}^{M_L} + \sum_{l=1}^{\infty} \Delta \boldsymbol{x}_{1.5\text{PN}}^{S_L} + \mathcal{O}(c^{-4}) \quad (20)$$

- important progress in the theory of light propagation:

S. Kopeikin, J. Math. Phys. **38** (1997) 2587

Solutions $\dot{\boldsymbol{x}}(t)$, $\boldsymbol{x}(t)$ highly involved. The question arises:
Which terms in (19) and (20) relevant for goal accuracy?

Total light deflection

From Eq. (11) one obtains the total light deflection

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \boldsymbol{\sigma} \times \left(\lim_{t_1 \rightarrow +\infty} \frac{\Delta \dot{\mathbf{x}}_{\text{PN}}(t_1)}{c} \times \boldsymbol{\sigma} \right) \quad (21)$$

- allows one to find relevant terms for given goal accuracy

$$\begin{aligned} \delta \left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{\hat{M}_L} \right) &= -\frac{4GM}{c^2 d_\sigma} \frac{(-1)^l}{(l-1)!} \hat{M}_L \hat{\partial}_L \ln |\boldsymbol{\xi}| \\ \delta \left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L} \right) &= -\frac{8G}{c^3} \frac{1}{|\boldsymbol{\xi}|} \epsilon_{abc} \sigma^c \frac{(-1)^l l^2}{(l+1)} \hat{S}_{bL-1} \hat{\partial}_{aL-1} \ln |\boldsymbol{\xi}| \end{aligned}$$

where $\hat{\partial}_L = \text{STF}_{i_1 \dots i_l} \frac{\partial}{\partial \xi^1} \cdots \frac{\partial}{\partial \xi^l}$ and $\boldsymbol{\xi} = \mathbf{d}_\sigma$

What is the magnitude of these terms?

- Time-Transfer Function (TTF) approach to get $\Delta \mathbf{x}_{1\text{PN}}^{M_L}$, $\Delta \mathbf{x}_{1.5\text{PN}}^{S_L}$
- represents another method in the theory of light propagation
- light travel time between two events: $(t_A, \mathbf{x}_A)$ and $(t_B, \mathbf{x}_B)$

$$t_A - t_B = \underbrace{\mathcal{T}(\mathbf{x}_A, \mathbf{x}_B)}_{\text{time-transfer-function}} \quad (22)$$

- applied to the case of axisymmetric bodies:

C. Le Poncine-Lafitte & P. Teyssandier, PRD **77** (2008) 044029

- total light deflection determined for axisymmetric bodies

$$\left| \delta \left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L} \right) \right| \leq \frac{4GM}{c^2} \frac{|J_l|}{d_\sigma} \left(\frac{P}{d_\sigma} \right)^l \quad (23)$$

- proof for $l = 2, 3, 4$, but validity for $l \geq 4$ was only conjectured
- no spin-multipoles

- total light deflection for axisymmetric bodies in rotation
- by using the approach of S. Kopeikin:

Total light deflection is related to Chebyshev polynomials

S. Zschocke: Physical Review D **107** (2023) 124055

$$\left| \delta \left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L} \right) \right| \leq \frac{4GM}{c^2} \frac{|J_l|}{d_\sigma} \left(\frac{P}{d_\sigma} \right)^l$$

$$\left| \delta \left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L} \right) \right| \leq \frac{8GM}{c^3} \Omega \frac{l^2}{l+4} |J_{l-1}| \left(\frac{P}{d_\sigma} \right)^{l+1}$$

- rigorous proof for all values of $l \geq 0$ in $\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\text{PN}}^{M_L})$
- rigorous proof for all values of $l \geq 1$ in $\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\text{PN}}^{S_L})$

3.5 Numerical values of total light deflection $[\mu\text{as}]$

Light deflection	Sun	Jupiter	Saturn
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1PN}}^{M_0}) $	1.75×10^6	16.3×10^3	5.8×10^3
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1PN}}^{M_2}) $	0.455	239	94
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1PN}}^{M_4}) $	0.008	9.6	5.41
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1PN}}^{M_6}) $	—	0.55	0.50
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1PN}}^{M_8}) $	—	0.04	0.06
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1PN}}^{M_{10}}) $	—	0.003	0.01
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1.5PN}}^{S_1}) $	0.7	0.17	0.04
$ \delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\text{1.5PN}}^{S_3}) $	—	0.026	0.008

- S. Zschocke: Physical Review D **107** (2023) 124055
 J_n of Sun: I.W. Roxburgh, A&A **377** (2001) 688.

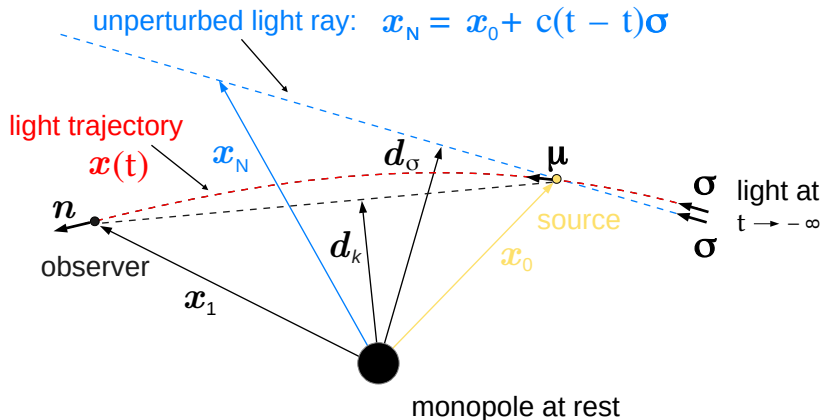
Some comments about 2PN calculations

- 2PN calculations only for monopole and quadrupole
- 2PN metric density: L. Blanchet, T. Damour (1986).
- 2PN metric: S. Zschocke, PRD **100** (2019) 084005

$$\begin{aligned}h_{00}^{(2\text{PM})} &= -\frac{2}{c^4} \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \frac{\hat{M}_L}{r} \right)^2 \\h_{0i}^{(2\text{PM})} &= 0 \\h_{ij}^{(2\text{PM})} &= +\frac{2}{c^4} \delta_{ij} \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \frac{\hat{M}_L}{r} \right)^2 \\&\quad - \frac{4}{c^4} \Delta^{-1} \left(\frac{\partial}{\partial x_i} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \frac{\hat{M}_L}{r} \right) \left(\frac{\partial}{\partial x_j} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \frac{\hat{M}_L}{r} \right)\end{aligned}$$

- Hadamard regularization necessary for the last term

3.6 2PN light propagation in field of monopole at rest



- first integration of geodesic equation in 2PN

$$\frac{\dot{\mathbf{x}}(t)}{c} = \boldsymbol{\sigma} + \frac{\Delta \dot{\mathbf{x}}_{1\text{PN}}^M}{c} + \frac{\Delta \dot{\mathbf{x}}_{2\text{PN}}^{M \times M}}{c}$$

- second integration of geodesic equation in 2PN

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta \mathbf{x}_{1\text{PN}}^M + \Delta \mathbf{x}_{2\text{PN}}^{M \times M}$$

2PN monopole solution determined at the first time by:

V. A. Brumberg: *Post-post Newtonian propagation of light in the Schwarzschild field*, Kinematica i fizika nebesnykh tel **3** (1987) 8

- 2PN monopole solution investigated in several articles
boundary value problem, PPN, transformations, etc ...
- recalling: upper limit of 1PN terms

$$\left| \delta \left(\mathbf{k}, \mathbf{n}_{1\text{PN}}^M \right) \right| \leq \frac{4 G M}{c^2} \frac{1}{d_k}$$

- upper limit of 2PN terms $\Rightarrow$ discovery of enhanced terms
 $\Rightarrow$ 2PN enhanced terms implemented in GREM

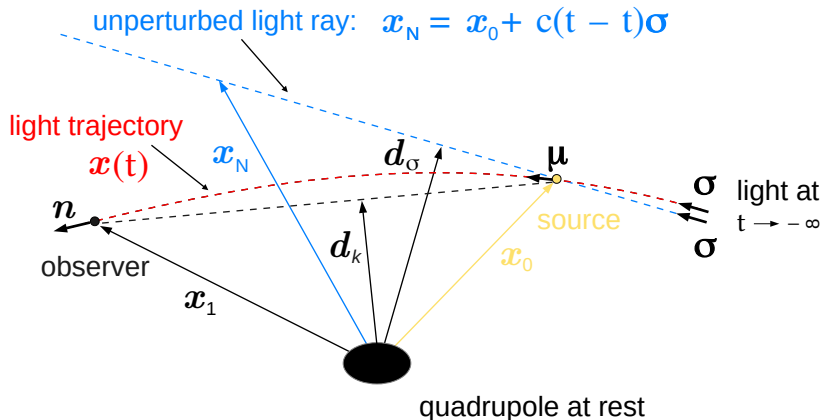
$$\left| \delta \left(\mathbf{k}, \mathbf{n}_{2\text{PN}}^{M \times M} \right) \right| \leq \frac{16 G^2 M^2}{c^4} \frac{1}{(d_k)^2} \frac{x_1}{d_k}$$

S.A. Klioner, S. Zschocke, CQG **27** (2010) 075015

N. Ashby, B. Bertotti, CQG **27** (2010) 145013

P. Teyssandier, C. Poncine-Lafitte, CQG **29** (2012) 245010

3.7 2PN light propagation in field of quadrupole at rest



- first integration of geodesic equation in 2PN

$$\frac{\dot{\mathbf{x}}(t)}{c} = \boldsymbol{\sigma} + \frac{\Delta \dot{\mathbf{x}}_{1\text{PN}}^M}{c} + \frac{\Delta \dot{\mathbf{x}}_{2\text{PN}}^{M \times M}}{c} + \frac{\Delta \dot{\mathbf{x}}_{2\text{PN}}^{M \times Q}}{c} + \frac{\Delta \dot{\mathbf{x}}_{2\text{PN}}^{Q \times Q}}{c}$$

- second integration of geodesic equation in 2PN

$$\mathbf{x}(t) = \mathbf{x}_N + \Delta \mathbf{x}_{1\text{PN}}^M + \Delta \mathbf{x}_{2\text{PN}}^{M \times M} + \Delta \mathbf{x}_{2\text{PN}}^{M \times Q} + \Delta \mathbf{x}_{2\text{PN}}^{Q \times Q}$$

- first determined by:

initial value: S. Zschocke: PRD **105** (2022) 024040

boundary value: S. Zschocke: PRD **111** (2025) 104069

- upper limit of 2PN terms $\implies$ discovery of enhanced terms

$$\begin{aligned} \left| \delta \left(\mathbf{k}, \mathbf{n}_{2\text{PN}}^{M \times M} \right) \right| &\leq \frac{16 G^2 M^2}{c^4} \frac{1}{(d_k)^2} \frac{x_1}{d_k} \\ \left| \delta \left(\mathbf{k}, \mathbf{n}_{2\text{PN}}^{M \times Q} \right) \right| &\leq \frac{64 G^2 M^2}{c^4} \frac{|J_2|}{(d_k)^2} \left(\frac{P}{d_k} \right)^2 \frac{x_1}{d_k} \\ \left| \delta \left(\mathbf{k}, \mathbf{n}_{2\text{PN}}^{Q \times Q} \right) \right| &\leq \frac{48 G^2 M^2}{c^4} \frac{|J_2|^2}{(d_k)^2} \left(\frac{P}{d_k} \right)^4 \frac{x_1}{d_k} \end{aligned}$$

- first determined by:
boundary value: S. Zschocke: PRD **112** (2025) 064087.

3.8 Numerical values of 2PN light deflection $[\mu\text{as}]$

Object	$ \delta(\mathbf{k}, \mathbf{n}_{2\text{PN}}^{M \times M}) $	$ \delta(\mathbf{k}, \mathbf{n}_{2\text{PN}}^{M \times Q}) $	$ \delta(\mathbf{k}, \mathbf{n}_{2\text{PN}}^{Q \times Q}) $
Jupiter	16.11	0.95	0.010
Saturn	4.42	0.29	0.003
Uranus	2.58	0.04	0.001
Neptune	5.83	0.08	0.002

3.9 Light propagation in 3PN approximation

- 3PN calculations are extremely rare:
- S. Zschocke: *A generalized lens equation for light deflection in weak gravitational fields*, CQG **28** (2011) 125016
- B. Linet, P. Teyssandier: *Enhanced term of order G^3 in the light travel time: Discussion for some solar system experiments*, arXiv:1312.3510 [gr-qc]

$$\left| \delta \left(\mathbf{k}, \mathbf{n}_{3\text{PN}}^M \right) \right| = \frac{128 G^3 M^3}{c^6} \frac{1}{(d_k)^3} \left(\frac{x_1}{d_k} \right)^2 \quad (24)$$

- this result has been found independently by:
Eq. (27) in S. Zschocke, CQG **28** (2011) 125016
Eqs. (21) and (22) in B. Linet, P. Teyssandier, arXiv:1312.3510
- numerical value: $12 \mu\text{as}$ in case of grazing light ray at Sun

4. Summary and Conclusions

What is known thus far?

- 1PN and 1.5PN solutions for full set of multipoles
- 2PN solutions are known for monopole and quadrupole
- 3PN upper limits are known for monopole

What is necessary for GREM on sub- μs level?

- mass-multipoles up to $l = 10$
- spin-multipoles up to $l = 3$
- 2PN solution for monopole and quadrupole
- most probably 2PN solution for octupole