

# The Rendering Equation

A Top-Down Approach

**Prof. Dr. Stefan Gumhold**

Chair of Computer Graphics and  
Visualization, TU Dresden



# Overview

## The Rendering Equation

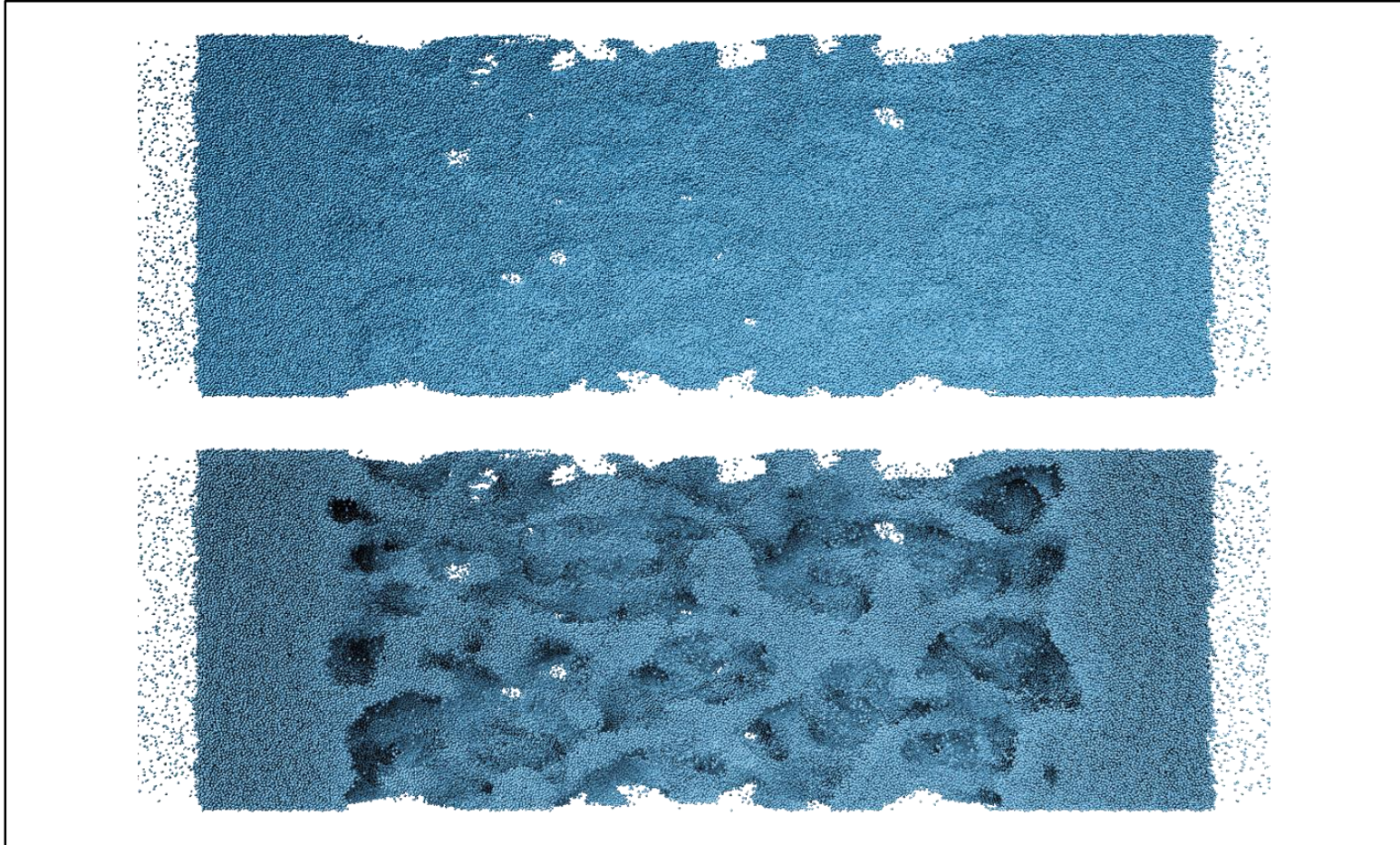
- ◆ motivation and simplifications
- ◆ light transport and rendering equation
- ◆ basic radiometry and hemispherical reflectance
- ◆ numerical solution strategies

## Literature

- ◆ [Kajiya](#), J. T. (1986). The Rendering Equation. Proc SIGGRAPH '86 Computer Graphics, 20(4), 143–150.
- ◆ Immel, D. S., Cohen, M. F., & Greenberg, D. P. (1986). A Radiosity Method for Non-Diffuse Environments. Proc. SIGGRAPH '86, Computer Graphics, 20(4), 133–142.

# Global Illumination

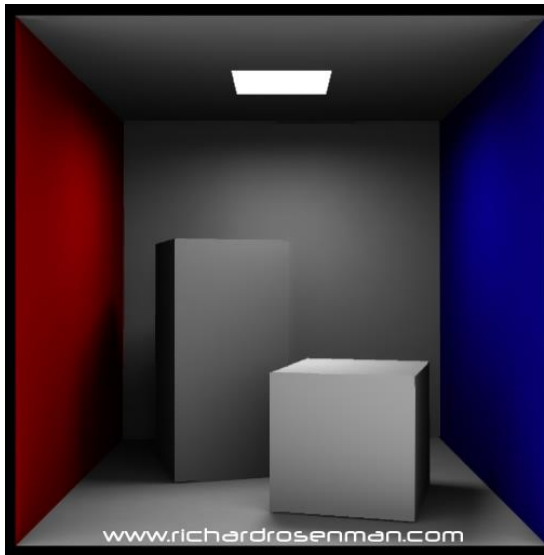
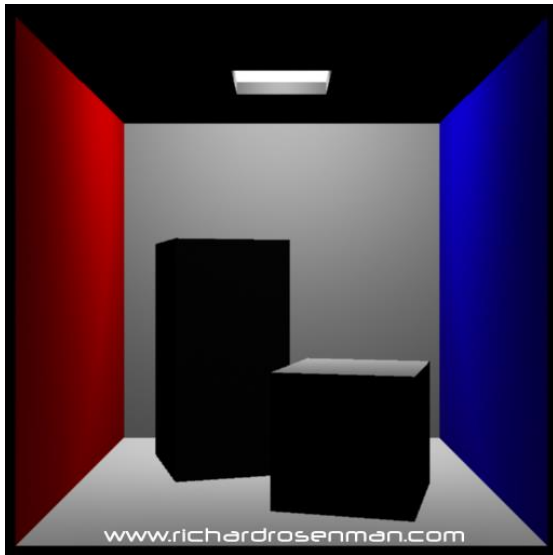
## Ambient Occlusion



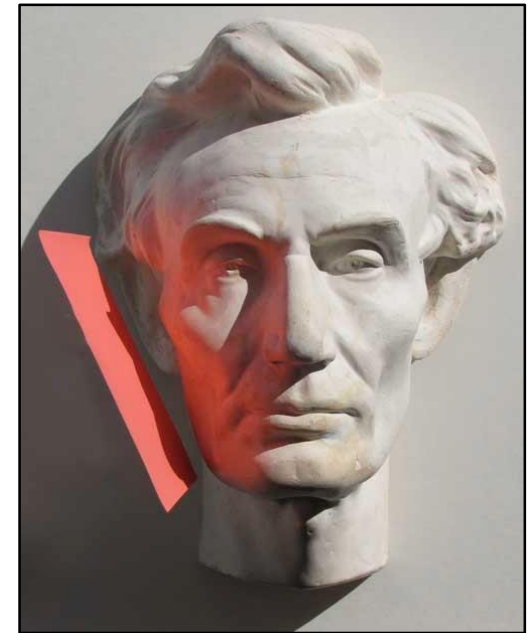
rendering a particle dataset without (top) and with (bottom) ambient occlusion  
© Sebastian Grottel

# Global Illumination

## Multiple Scattering



increasing number of  
diffuse surface scattering:  
top left: *once*  
top right: *twice*  
bottom left: *infinite*



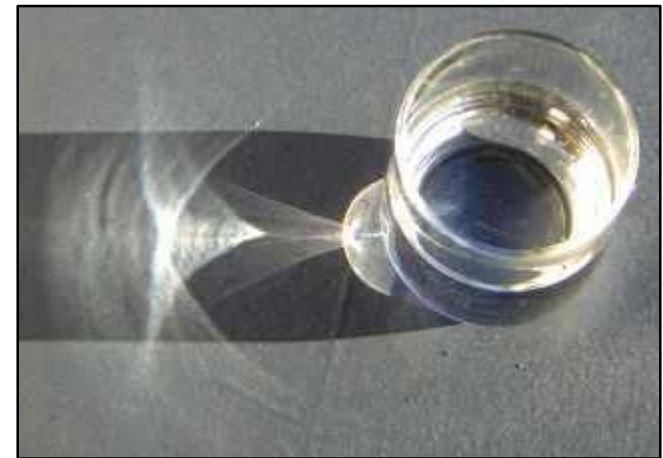
example of color bleeding  
© James Gurney

# Global Illumination

## Mirror Reflection and Caustics



example of mirror reflections and soft shadows  
© Prairie Light Images



example of caustic where light is bundled by concave curved refractive or mirror reflective surfaces, © Heiner Otterstedt



Crepuscular Rays are volumetric scattering effects for back light illumination  
© Wikipedia



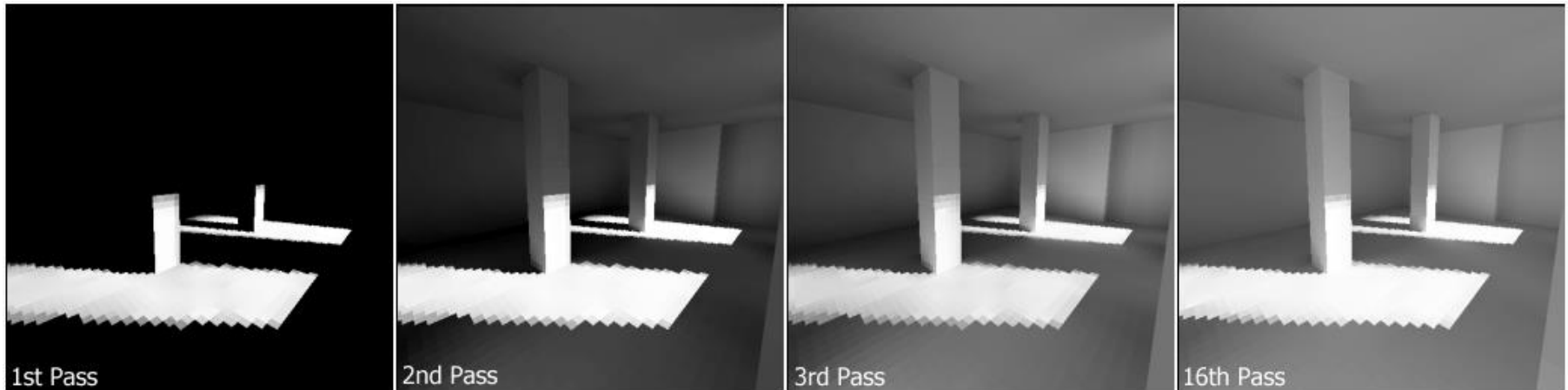
Realtime volumetric scattering in game engine,  
© Markus Billeter et al 2012



# Overview

## The Rendering Equation

- ◆ motivation and simplifications
- ◆ light transport and rendering equation
- ◆ basic radiometry and hemispherical reflectance
- ◆ numerical solution strategies



© Wikipedia – Radiosity, [link](#)

- ◆ light is reflected between mutually visible surfaces
- ◆ objects reflect and become emitters too
- ◆ **dynamic energy equilibrium**

emitted photons = absorbed photons + escaped photons

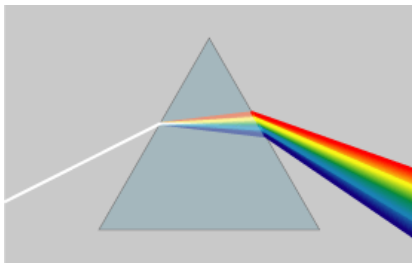
# Light Transport

## Setting for Opaque Scenes

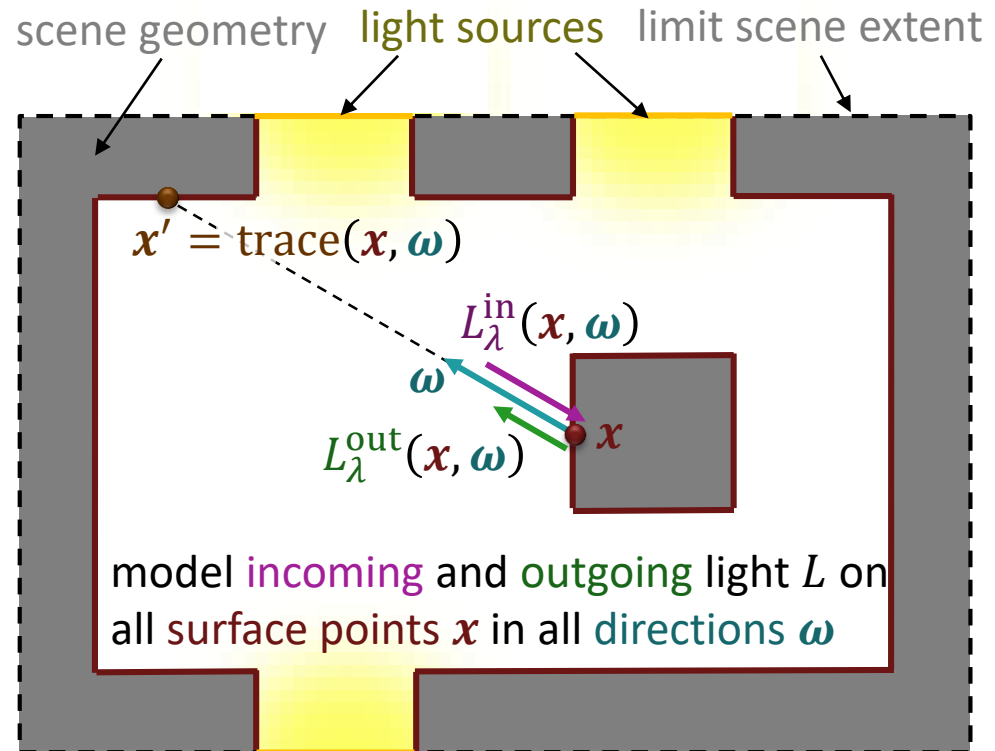


### Typical Simplifications

- macroscopic
- geometric optics (linear propagation)
- no volumetric scattering
- no dispersion



- ignore polarization



linking of incoming and outgoing light with raytracing operator:

$$L_{\lambda}^{\text{in}}(x, \omega) = L_{\lambda}^{\text{out}}(\text{trace}(x, \omega), -\omega)$$



- total outgoing light is emission plus reflection

$$L_{\lambda}^{\text{out}} = L_{\lambda}^{\text{emit}} + L_{\lambda}^{\text{reflect}}$$

$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) + \underline{L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}})}$$

- reflect incoming light from all directions with BRDF  $\rho$

$$\underline{L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}})} = \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$$



- link outgoing to incoming light with trace

$$L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) = L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}), -\boldsymbol{\omega}^{\text{in}})$$

$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) +$$

$$\iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}), -\boldsymbol{\omega}^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$$

directional formulation of *rendering equation*, which is also called *transport equation*

# Rendering Equation

## Directional Form [Immel et al 1986]



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) + \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}), -\boldsymbol{\omega}^{\text{in}}) \cos\theta^{\text{in}} d\Omega^{\text{in}}$$

directional formulation of *rendering equation*, which is also called *transport equation*

### Parameters

- $\mathbf{x}$ ... surface point
- $\boldsymbol{\omega}^{\text{out}}$ ... outgoing direction
- $\boldsymbol{\omega}^{\text{in}}$ ... incoming direction
- $\theta^{\text{in}}$ ... angle between surface normal and incoming direction
- $\Omega^{\text{in}}$ ... outer hemisphere of surface is integration domain for incoming directions

### Functions

- $L_{\lambda}^{\text{out}}$ ... total outgoing radiance and to be solved for (*[Fredholm integral equation of 2nd kind](#)*)
- $L_{\lambda}^{\text{emit}}$ ... emitted radiance
- $\rho$ ... bidirectional reflectance distribution function (BRDF)
- $\text{trace}$ ... raytracing operator

# Rendering Equation

## Important Topics related to RenEq



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \omega^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \omega^{\text{out}}) + \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \omega^{\text{in}}), -\omega^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$$

### radiometry

- How to parametrize and integrate directions  $\omega$ ?
- How to define  $L_{\lambda}(\mathbf{x}, \omega)$  and  $\rho(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}})$ ?

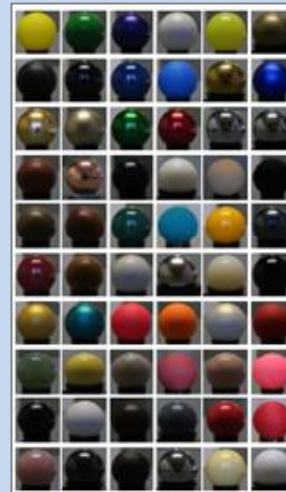
### photometry

- How do we perceive light?

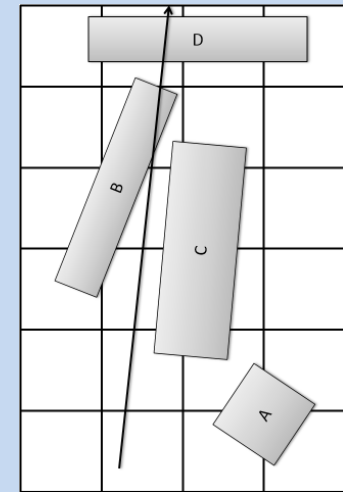
### modeling light sources



### modeling materials



### ray-scene-intersection



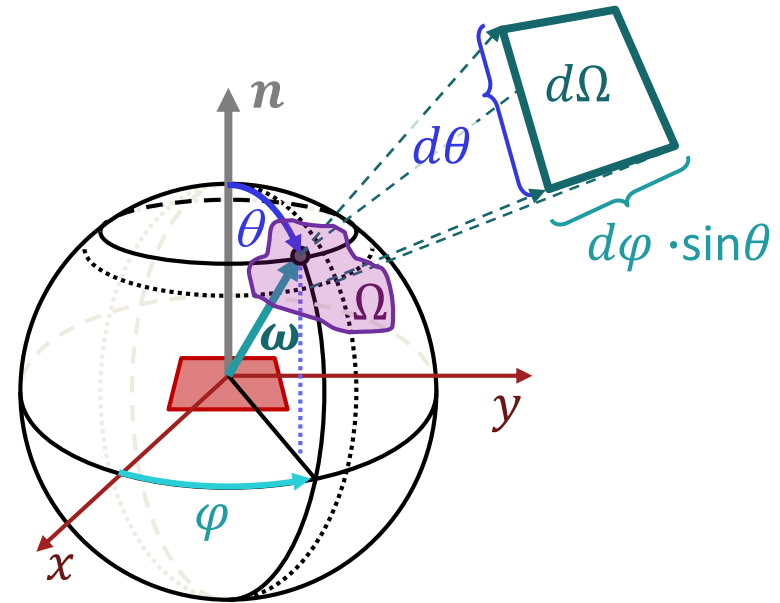
# Directions and Solid Angle

## Local Parametrization & Integration



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) + \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos\theta^{\text{in}} d\Omega^{\text{in}}$$

- direction  $\boldsymbol{\omega}$  is parametrized over *unit sphere* in spherical coordinates  $\varphi, \theta$
- parametrization is relative to local surface normal  $\mathbf{n}$
- solid angle  $\Omega$  is measured in area covered on sphere
- SI unit: *sr* (steradian)
- integration of solid angle yields double integral over  $\varphi$  and  $\theta$



$$\boldsymbol{\omega} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}, \quad \begin{aligned} \varphi &\in [-\pi, \pi] \\ \theta &\in [0, \frac{\pi}{2}] \end{aligned}$$

$$d\Omega = d\varphi \cdot \sin\theta d\theta$$

# Directions and Solid Angle

## Example Integration



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) + \iint_{\Omega^{\text{in}}} \underbrace{\rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}})}_C \cos\theta^{\text{in}} d\Omega^{\text{in}}$$

$$\begin{aligned} \iint_{\Omega^{\text{in}}} C \cos\theta^{\text{in}} d\Omega^{\text{in}} &= C \int_0^{\frac{\pi}{2}} \left( \int_{-\pi}^{\pi} \cos\theta^{\text{in}} d\varphi \right) \cdot \sin\theta^{\text{in}} d\theta^{\text{in}} \\ &= C \int_0^{\frac{\pi}{2}} \left( \int_{-\pi}^{\pi} d\varphi \right) \cos\theta^{\text{in}} \cdot \sin\theta^{\text{in}} d\theta^{\text{in}} \\ &= 2\pi C \int_0^{\frac{\pi}{2}} \cos\theta^{\text{in}} \cdot \sin\theta^{\text{in}} d\theta^{\text{in}} \\ &= \pi C \left[ -\cos^2\theta^{\text{in}} \right]_0^{\frac{\pi}{2}} = \pi C \end{aligned}$$

# Directions and Solid Angle

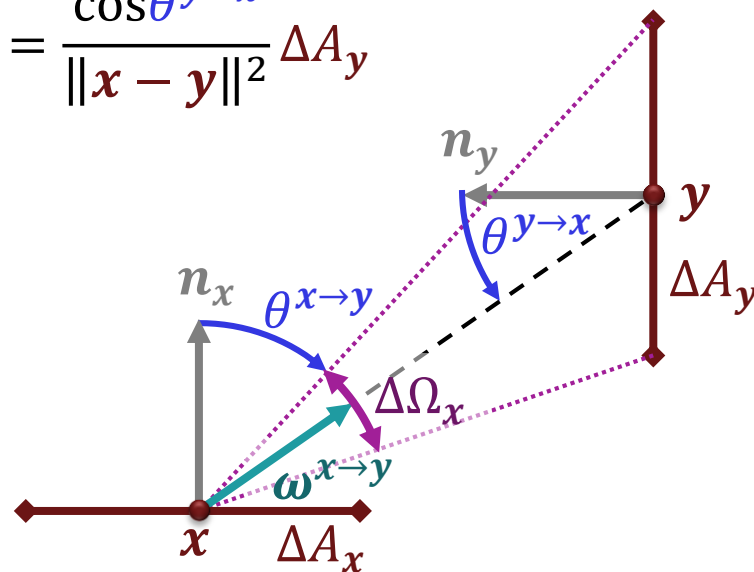
## Connection to Surface Patches



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \omega^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \omega^{\text{out}}) + \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \omega^{\text{in}}) \underline{\cos \theta^{\text{in}} d\Omega^{\text{in}}}$$

- patch  $\Delta A_y$  is seen from patch  $\Delta A_x$  under the solid angle  $\Delta \Omega_x$  that decreases with distance squared and cosine to normal of patch around  $\mathbf{y}$
- plugging into rendering equation yields new form

$$\Delta \Omega_x = \frac{\cos \theta^{y \rightarrow x}}{\|\mathbf{x} - \mathbf{y}\|^2} \Delta A_y$$



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \omega^{\text{out}}) = L_{\lambda}^{\text{emit}} + \iint_{y \in A} \rho(\mathbf{x}, \omega^{x \rightarrow y}, \omega^{\text{out}}) L_{\lambda}^{\text{out}}(\mathbf{y}, \omega^{y \rightarrow x}) V(\mathbf{x}, \mathbf{y}) \frac{\cos \theta^{x \rightarrow y} \cos \theta^{y \rightarrow x}}{\|\mathbf{x} - \mathbf{y}\|^2} dA_y$$

*rendering equation*, with integration over scene area and visibility check

# Rendering Equation II

## Area Formulation [Kajiya et al 1986]



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \omega^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \omega^{\text{out}}) + \iint_{\mathbf{y} \in A} \rho(\mathbf{x}, \omega^{\mathbf{x} \rightarrow \mathbf{y}}, \omega^{\text{out}}) L_{\lambda}^{\text{out}}(\mathbf{y}, \omega^{\mathbf{y} \rightarrow \mathbf{x}}) G(\mathbf{x}, \mathbf{y}) dA_{\mathbf{y}}$$

area formulation of *rendering equation*

- integration of  $\mathbf{y}$  over complete scene area  $A$
- geometry factor  $G(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) \frac{\cos \theta^{\mathbf{x} \rightarrow \mathbf{y}} \cos \theta^{\mathbf{y} \rightarrow \mathbf{x}}}{\|\mathbf{x} - \mathbf{y}\|^2}$
- when combining  $G$  with  $dA_{\mathbf{y}}$  the units combine to sr:  
$$[G \cdot dA_{\mathbf{y}}] = \frac{1}{m^2} \cdot m^2 = \frac{m^2}{m^2} = sr.$$
- with visibility test  $V(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \dots \text{if visible} \\ 0 & \dots \text{if blocked} \end{cases}$

# Rendering Equation III

## Operator Formulation



- ◆ To gain insight on a higher level, one can define ray **tracing** and **reflection** as **functionals** or **operators**  **$T$**  and  **$R$**  that map radiance functions onto radiance functions:
- ◆  **$T$**  :  $L_{\lambda}^{\text{out}} \rightarrow L_{\lambda}^{\text{in}}$ ,  $L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}) = L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}), -\boldsymbol{\omega})$
- ◆  **$R$**  :  $L_{\lambda}^{\text{in}} \rightarrow L_{\lambda}^{\text{reflect}}$ ,  $L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos\theta^{\text{in}} d\Omega^{\text{in}}$
- ◆ One can interpret the action of operators on functions like infinite dimensional matrices acting on infinite dimensional vectors, yielding the **operator form** of the rendering equation:

$$L_{\lambda}^{\text{out}} = L_{\lambda}^{\text{emit}} + \mathbf{R} \cdot \mathbf{T} \cdot L_{\lambda}^{\text{out}}$$

- ◆ solving for  $L_{\lambda}^{\text{out}}$  formally with Taylor expansion yields
$$L_{\lambda}^{\text{out}} = (\mathbf{I} - \mathbf{R} \cdot \mathbf{T})^{-1} L_{\lambda}^{\text{emit}} = L_{\lambda}^{\text{emit}} + \mathbf{RT} \cdot L_{\lambda}^{\text{emit}} + (\mathbf{RT})^2 \cdot L_{\lambda}^{\text{emit}} + \dots$$
- ◆ Thus the radiance function can be computed as the sum over emitted light, single reflection of emitted light, double reflection of emitted light and so on...

# Rendering Equation III

## Operator Formulation Refined

- According to the splitting of the BRDF into diffuse ( $\mathbf{D}$ ), specular ( $\mathbf{S}$ ) and mirror reflecting ( $\mathbf{M}$ ) part, one can refine the combined trace-reflection operator:

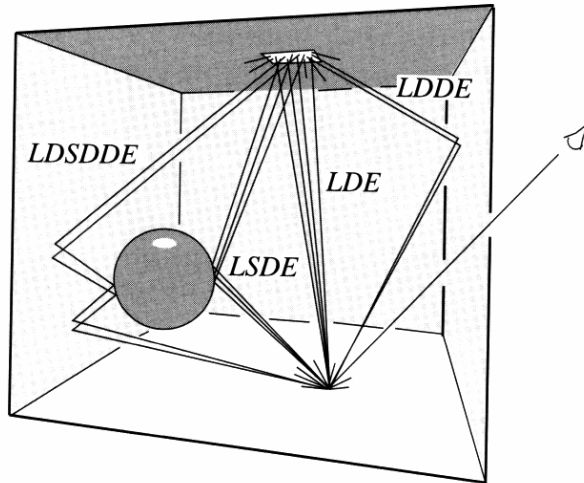
$$\mathbf{RT} = \mathbf{D} + \mathbf{S} + \mathbf{M}$$

- Refinement of formal solution yields

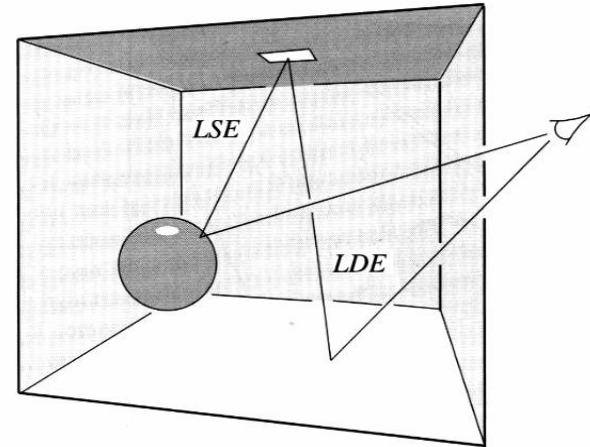
$$L_{\lambda}^{\text{out}} = L_{\lambda}^{\text{emit}} + (\mathbf{D} + \mathbf{S} + \mathbf{M}) \cdot L_{\lambda}^{\text{emit}} + (\mathbf{D} + \mathbf{S} + \mathbf{M})^2 \cdot L_{\lambda}^{\text{emit}} + \dots$$

- If  $L_{\lambda}^{\text{out}}$  is received in eye, this means that the rendering equation is solved by adding up contributions by all different kinds of ray paths.
- The paths are typically denoted in order of light transport from light source ( $L$ ) to eye ( $E$ ):  $LDDSM E$
- One uses regular expression syntax to capture sets of rays:  $L(D|S)^*E$

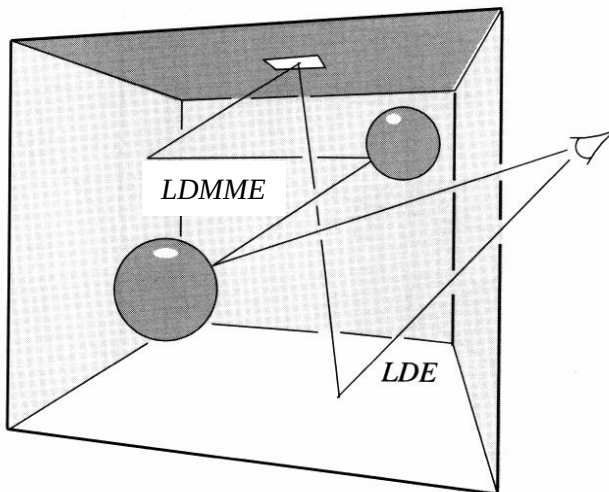
# Operator Formulation Examples



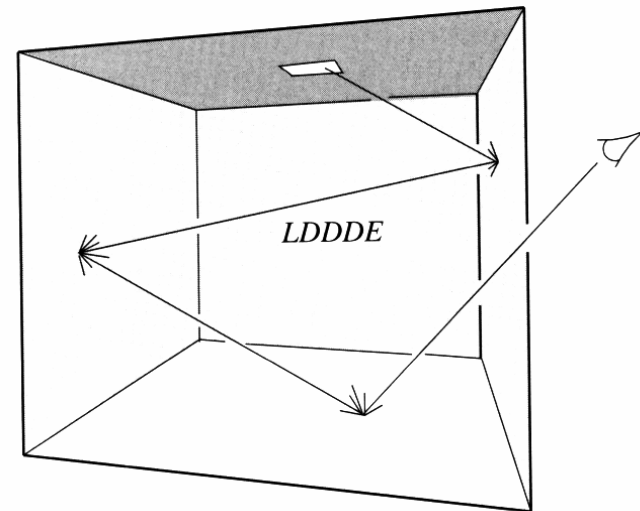
all paths:  $L(D|S|M)*E$



local illumination:  $L(D|S)E$



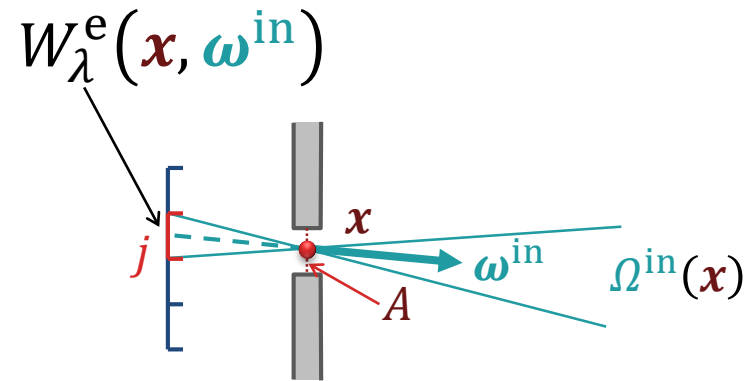
raytracing:  $L(S|D)M*E$



radiosity:  $LD*E$

# Sensor Efficiency

- dual to light sources the human eye or a camera acts as light sensor
- The function  $W_\lambda^e(\mathbf{x}, \omega^{\text{in}})$  defines the **spectral efficiency** of the sensor for an incoming ray.
- Efficiency is the percentage of radiance, which was detected
- In the simplest case efficiency is one for rays that hit the sensor and zero otherwise.
- The brightness of the pixel is spectral power and computed through the **measurement equation** as integral over all incoming rays hitting the sensor.



simplified camera model with sample point  $\mathbf{x}$  on aperture of area  $A$  and solid angle of incoming directions that hit sample pixel  $j$

$$\Phi_{\lambda,j} = \iint_A \iint_{\Omega_j^{\text{in}}(\mathbf{x})} W_\lambda^e(\mathbf{x}, \omega^{\text{in}}) L_\lambda^{\text{in}}(\mathbf{x}, \omega^{\text{in}}) \cos\theta^{\text{in}} d\Omega_j^{\text{in}} dA$$

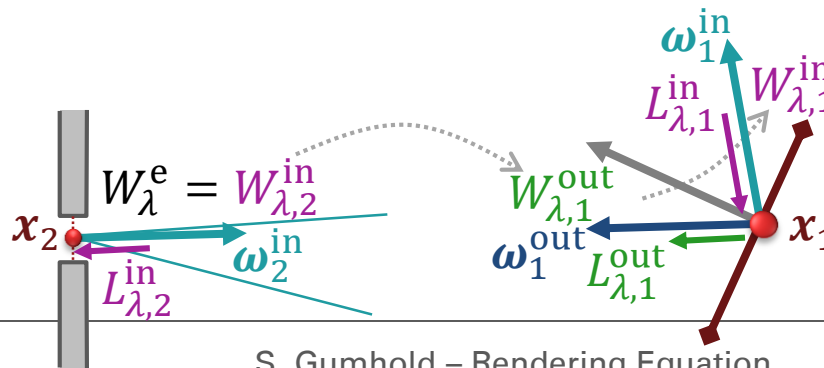
measurement equation

# Importance Function

- To steer the integration of the rendering equation one defines the spectral **incoming** and **outgoing importance function** (also called **potential** or **response function**)  $W_\lambda^{\text{in}} / W_\lambda^{\text{out}}$ .
- $W_\lambda^{\text{in}} / W_\lambda^{\text{out}}$  measures the influence of the radiance  $L_\lambda^{\text{in}} / L_\lambda^{\text{out}}$  onto the sensor (often restricted to one pixel)
- based on the sensor efficiency function  $W^e$  importance is computed as follows and flows from the sensor to the light sources

$$W_\lambda^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) = W_\lambda^e(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) + \iint_{\Omega^{\text{out}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}, \boldsymbol{\omega}^{\text{in}}) W_\lambda^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}}$$

spectral importance function  $W_\lambda^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = W_\lambda^{\text{in}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}), -\boldsymbol{\omega}^{\text{out}})$





- ◆ The rendering equation combines light transport and equilibrium of light emission and absorption processes
- ◆ There are two parameterizations of the rendering equation: over directions and over surface area
- ◆ The operator notation allows to formally solve the rendering equation by adding contributions of all types of light paths
- ◆ Sensor efficiency is defined per ray that hits sensors
- ◆ Measurements are integrated over ray space and measured in units of power
- ◆ Importance function is used to measure influence of radiance along an arbitrary ray in the scene on the measurement with a sensor



# Overview

## The Rendering Equation

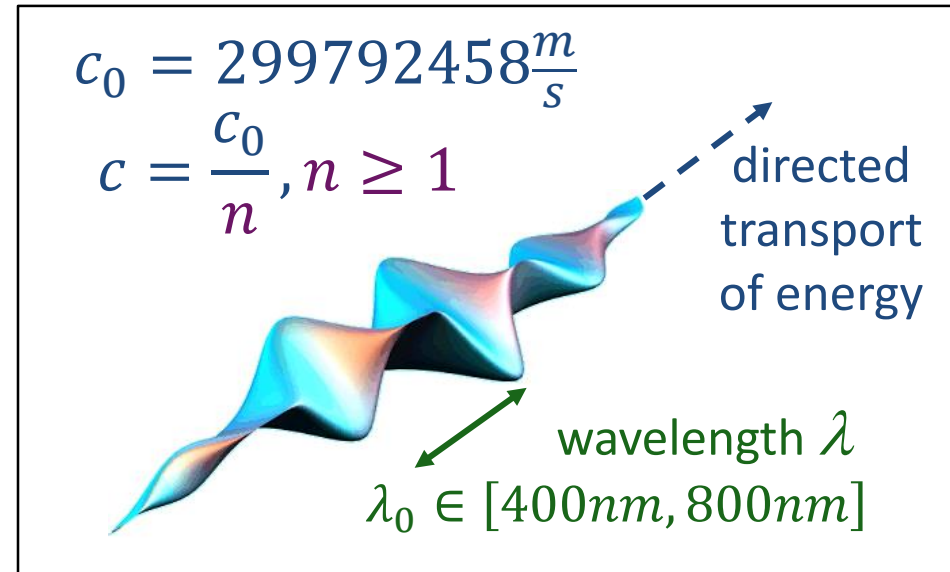
- ◆ motivation and simplifications
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# Radiometry

## Basic Properties of Light



- light is composed of photons
- photons travel with the speed of light  $c$
- in medium slower by index  $n$  of refraction
- each photon has characteristic frequency  $\nu$  and wavelength  $\lambda$
- each photon carries a tiny amount of energy  $q$ , characterized by frequency



visualization of a light photon, © 1988 by the  
TAT Foundation, [link](#)

$$c = \nu \cdot \lambda$$

$$\nu \in [3.75 \cdot 10^{14} \text{Hz}, 7.5 \cdot 10^{14} \text{Hz}]$$

$$q = h \cdot \nu, h = 6.626 \cdot 10^{-34} \text{Js}$$

$$q \in [2.5 \cdot 10^{-19} \text{J}, 5.0 \cdot 10^{-19} \text{J}]$$

# Radiometry

## Spectral Energy Density



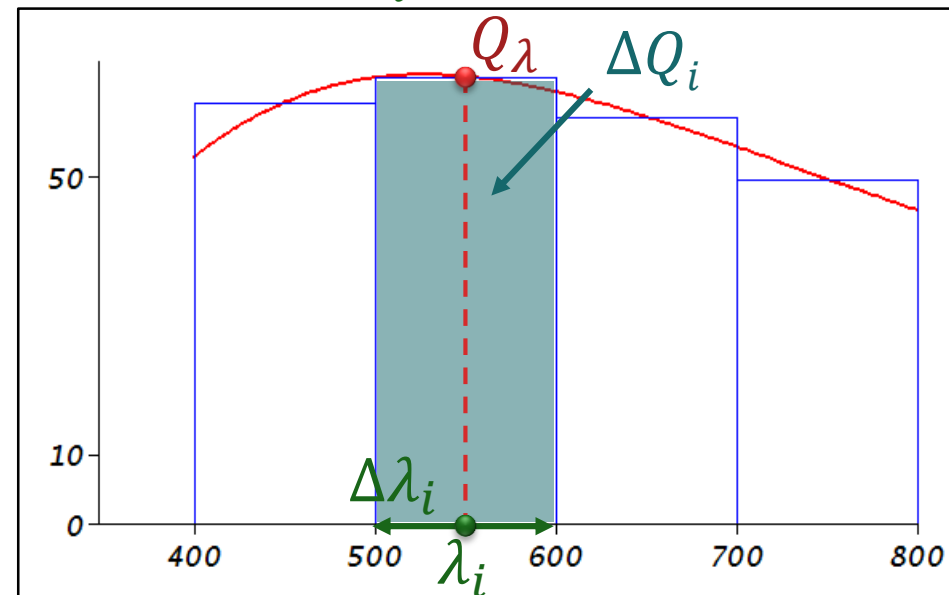
- spectral filter allows to measure photon energy  $\Delta Q_i$  within wavelength interval  $\Delta\lambda_i$  around  $\lambda_i$
- relating  $\Delta Q_i$  to the interval width yields the spectral energy density  $Q_\lambda(\lambda_i)$
- with narrower filters spectral energy density converges to derivative

$$Q_\lambda(\lambda) = \frac{dQ}{d\lambda}$$

$$Q = \int_{400nm}^{800nm} Q_\lambda(\lambda) d\lambda \approx \sum_i \Delta Q_i \cdot \Delta\lambda_i$$

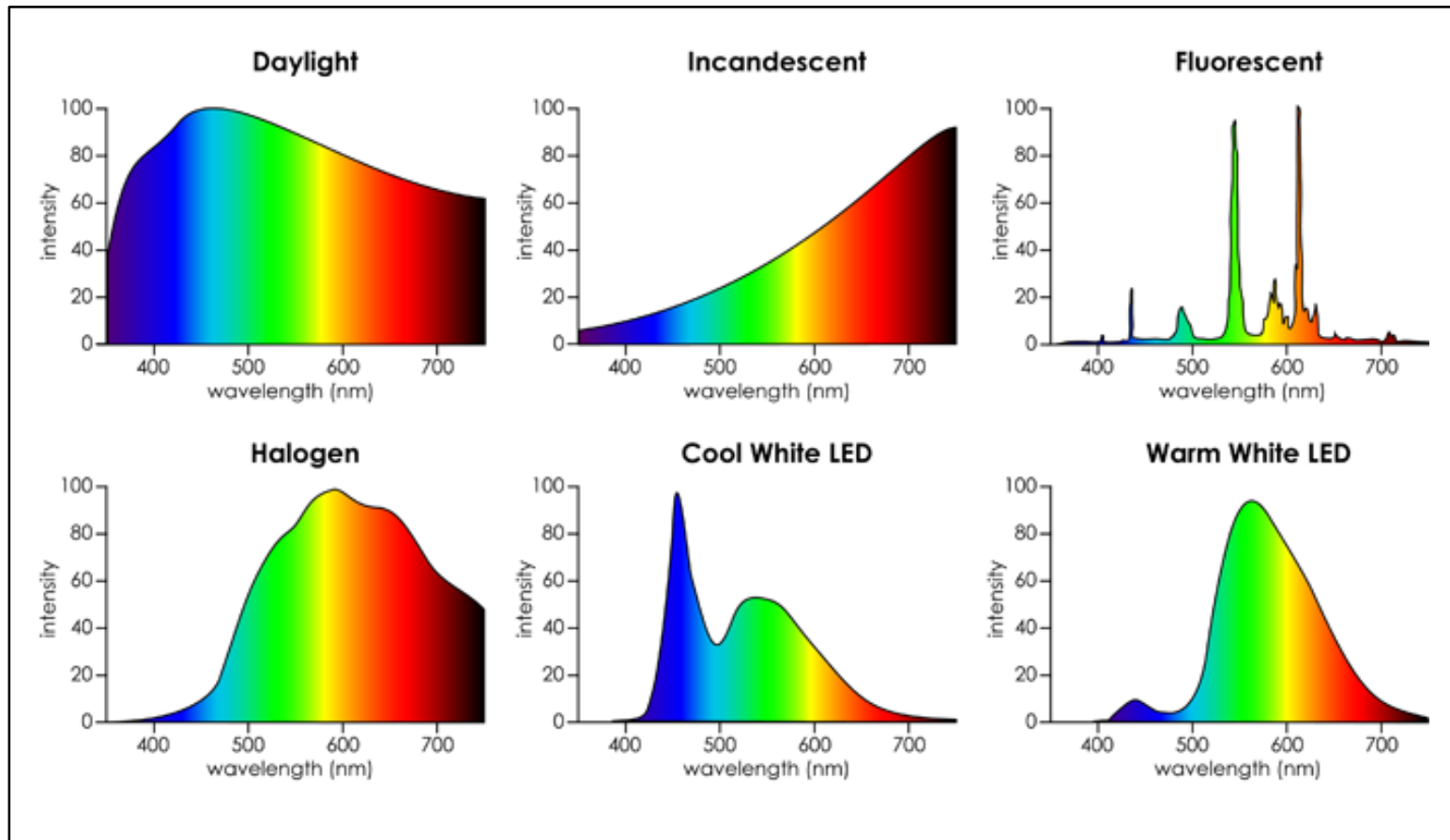
$$\Delta Q_i = \sum_{j|\lambda_j \in \Delta\lambda_i} q_j$$

$$Q_\lambda(\lambda_i) \approx \frac{\Delta Q_i}{\Delta\lambda_i}$$



# Radiometry

## Spectral Energy Density



emission spectrum of different light sources, © [housecraft](#)

# Radiometry

## Power and Irradiance



- spectral power  $\Phi_\lambda$  is density of spectral energy  $Q_\lambda$  in time  $t$

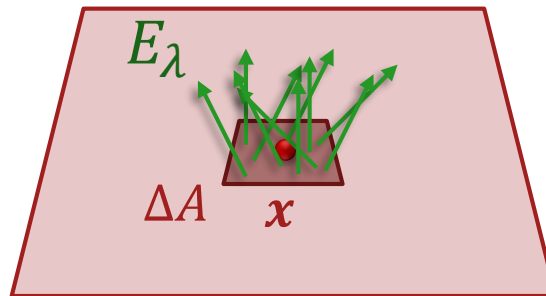
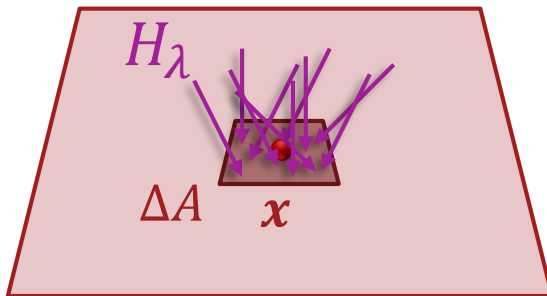
$$\Phi_\lambda(\lambda, t) = \frac{dQ_\lambda}{dt}$$

$$Q_\lambda(\lambda) = \int \Phi_\lambda(\lambda, t) dt \approx \sum \Phi_{\lambda,i} \Delta t_i$$

- irradiance  $H_\lambda$  / radiant exitance or radiosity  $E_\lambda$  are densities of incoming / emitted spectral power in surface area  $A$

$$H_\lambda | E_\lambda(\mathbf{x}) = \frac{d\Phi_\lambda^{\text{in|out}}}{dA}$$

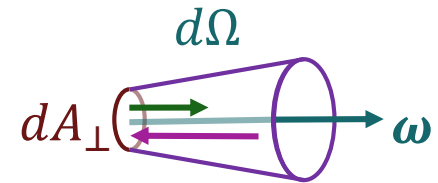
$$\Phi_\lambda^{\text{in}} = \iint H_\lambda(\mathbf{x}) dA \approx \sum H_{\lambda,i} \Delta A_i$$



$$dA = dx \cdot dy$$

$$\Delta A = \Delta x \cdot \Delta y$$

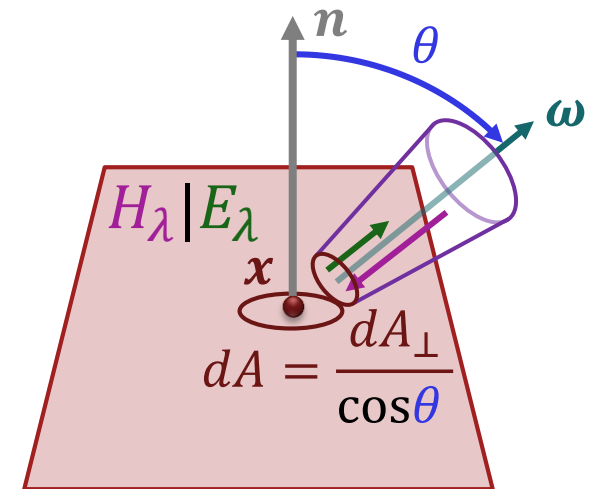
- incoming and outgoing radiance  $L_\lambda^{\text{in|out}}$  are densities of  $\Phi_\lambda^{\text{in|out}}$  over the direction  $\omega$  of transport and the area  $A_\perp$  orthogonal to  $\omega$ .



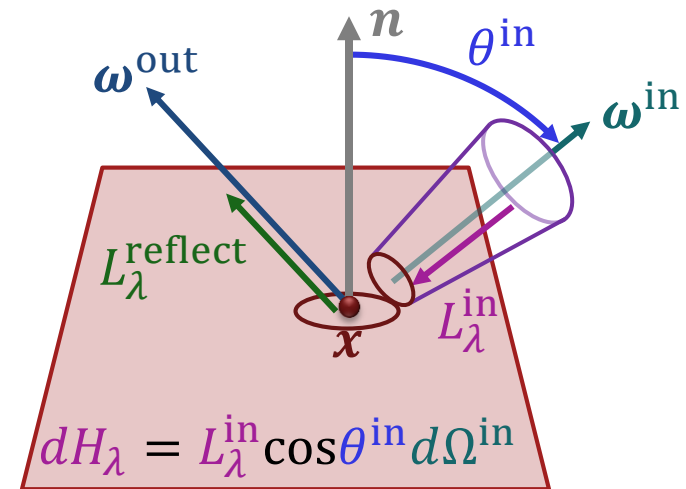
$$L_\lambda^{\text{in|out}}(\mathbf{x}, \omega) = \frac{d^2 \Phi_\lambda^{\text{in|out}}}{dA_\perp d\Omega^{\text{in|out}}}$$

- in vacuum *radiance is constant along ray*
- carfull:** relating to surface area needs additional cosine term:

$$H_\lambda | E_\lambda(\mathbf{x}) = \frac{d\Phi_\lambda}{dA} = \frac{\cos\theta \cdot d\Phi_\lambda}{dA_\perp} = \iint_{\Omega} L_\lambda(\mathbf{x}, \omega) \cos\theta d\Omega$$



- surface reflection distributes **incoming light** over all **outgoing** directions
- the bidirectional reflectance distribution function  $\rho$  (BRDF) is density of **reflected radiance**  $L_{\lambda}^{\text{reflect}}$  with respect to **irradiance**  $H_{\lambda}$
- integration over hemisphere yields total reflected radiance



$$\rho_{\lambda}(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) = \frac{dL_{\lambda}^{\text{reflect}}(\mathbf{x}, \omega^{\text{out}})}{dH_{\lambda}(\mathbf{x}, \omega^{\text{in}})}$$

$$L_{\lambda}^{\text{reflect}}(\mathbf{x}, \omega^{\text{out}}) = \iint_{\Omega^{\text{in}}} \rho_{\lambda}(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \omega^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$$

# Radiometry

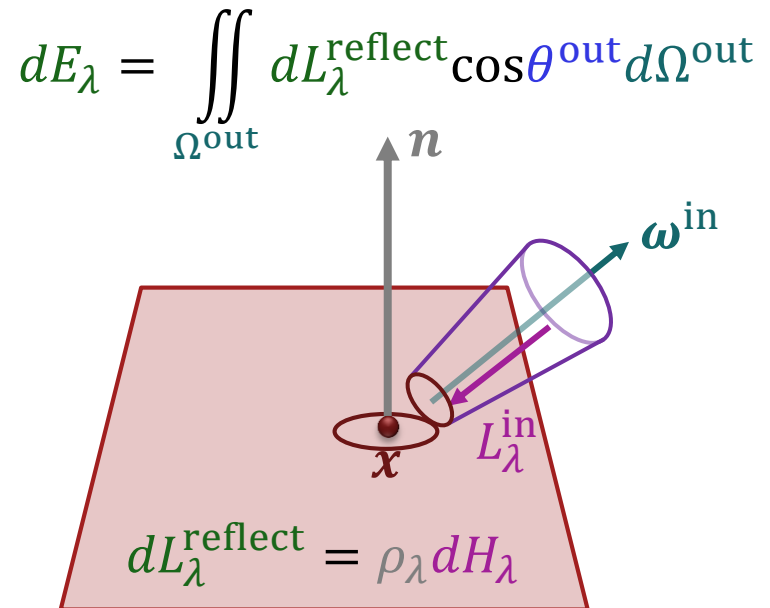
## Diffuse BDRF & Reflectance



- directional hemispherical reflectance  $R_\lambda$  or albedo computes fraction of outgoing light over incoming light focused around  $\omega^{\text{in}}$

$$R_\lambda(\omega^{\text{in}}) = \frac{dE_\lambda(\omega^{\text{in}})}{dH_\lambda(\omega^{\text{in}})} \in [0,1]$$

$$= \iint_{\Omega^{\text{out}}} \rho_\lambda(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) \cos\theta^{\text{out}} d\Omega^{\text{out}}$$



- for a constant (diffuse) BRDF, also  $R_\lambda$  is constant

$$R_\lambda(\omega^{\text{in}}) = \rho_\lambda^{\text{diff}} \iint_{\Omega^{\text{out}}} \cos\theta^{\text{out}} d\Omega^{\text{out}} = \pi \rho_\lambda^{\text{diff}}$$

→  $R_\lambda$  defines  $\rho_\lambda^{\text{diff}}$  fully:

$$\rho_\lambda^{\text{diff}} = R_\lambda / \pi$$

# Radiometry

## Diffuse BDRF & Reflectance



- Full derivation of formula for directional hemispherical reflectance

$$E_{\lambda}(\mathbf{x}) = \iint_{\Omega^{\text{out}}} L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}}$$

$$L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = \iint_{\Omega^{\text{in}}} \rho_{\lambda}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$$

$$E_{\lambda}(\mathbf{x}) = \iint_{\Omega^{\text{in}}} \left[ \iint_{\Omega^{\text{out}}} \rho_{\lambda}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}} \right] \underbrace{L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}}_{dH_{\lambda}}$$

$$E_{\lambda}(\mathbf{x}) = \iint_{\Omega^{\text{in}}} \left[ \iint_{\Omega^{\text{out}}} \rho_{\lambda}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}} \right] dH_{\lambda}$$

$$R_{\lambda}(\boldsymbol{\omega}^{\text{in}}) = \frac{dE_{\lambda}(\boldsymbol{\omega}^{\text{in}})}{dH_{\lambda}(\boldsymbol{\omega}^{\text{in}})} = \iint_{\Omega^{\text{out}}} \rho_{\lambda}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}}$$



- ◆ radiometric quantities are derived from photon energy per time (power) and the power densities with respect to area and or direction
- ◆ most important is radiance – the light power travelling along a ray. Without volumetric scattering, the radiance is constant along rays.
- ◆ The BRDF relates incoming irradiance with outgoing radiance and is integrated over hemisphere relative to surface normal.
- ◆ BRDFs can be analyzed by integrating over one directional argument. Integration over outgoing directions yields directional hemispherical reflections that must be  $\leq 1$ .



# Overview

## The Rendering Equation

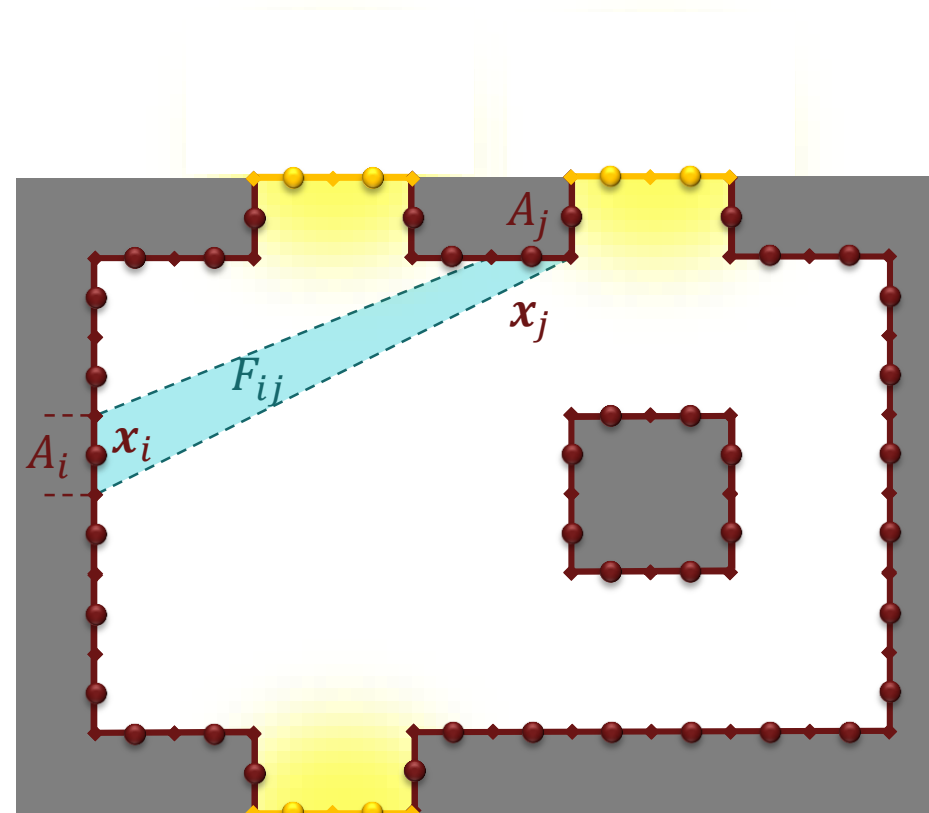
- ◆ motivation and simplifications
- ◆ light transport and rendering equation
- ◆ basic radiometry and hemispherical reflectance
- ◆ numerical solution strategies

# Solving the Rendering Equation

## Radiosity



- ◆ **Assumption:** only **diffuse** emission and reflection
- ◆ dependency on direction can be dropped on outgoing radiance  $L_{\text{out}}$  and restriction to exitant radiosity  $E$  possible
- ◆ discretize surface into patches  $A_i$  around  $\mathbf{x}_i$
- ◆ integrate pairwise geometry terms over patches:  $F_{ij}$
- ◆ derive system of linear equations:  
$$E_{\lambda,i}^{\text{out}} = E_{\lambda,i}^{\text{emit}} + \sum_j \rho_i F_{ij} E_{\lambda,j}^{\text{out}}$$
- ◆ and solve for vector of radiosities  $E_{\lambda,i}^{\text{out}}$
- ◆ solution is independent of view point and can be baked



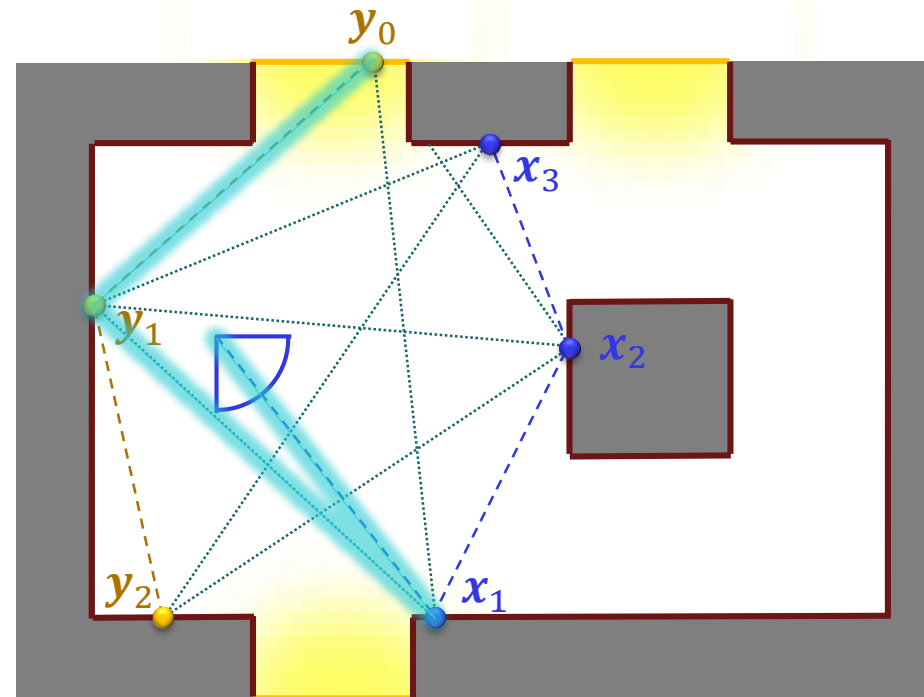
# Solving the Rendering Equation

## Bidirectional Path-Tracing



- define camera and repeat multiple times per pixel
- sample camera path
  - sample primary ray and find intersection  $x_1$
  - while Russian roulette
    - sample secondary ray and find intersection  $x_i$
- sample light path
  - sample light position  $y_0$
  - while Russian roulette
    - sample secondary ray and find intersection  $y_i$
- interconnect paths

### Monte-Carlo Integration



compare java-script Path-Tracing under  
<http://madebyevan.com/webgl-path-tracing>  
and try to find a caustic

# Summary

- ◆ light is modeled as continuous radiance fields on scene surface
- ◆ light bounces between visible surfaces until equilibrium
- ◆ bounces are described by BRDF (material property)
- ◆ rendering equation is integral equation in outgoing radiance (or optionally in incoming radiance)
- ◆ pixel values are computed by integration of radiance over the rays hitting the sensor
- ◆ solution strategies build on discretizations of the scene or monte-carlo integration techniques
- ◆ assumption of pure diffuse emission and reflection reduces problem to a system of linear equations



# References

## ◆ Books

- ◆ Philip Dutre; Philippe Bekaert; Kavita Bala, [Advanced Global Illumination, 2nd Edition](#), *chapter 2*
- ◆ Matt Pharr; Greg Humphreys, [Physically Based Rendering, 2nd Edition](#), *chapter 5, 15*
- ◆ Peter Shirley; Michael Ashikhmin; Steve Marschner, [Fundamentals of Computer Graphics](#), *chapter 20, 24*
- ◆ John F. Hughes; Andries van Dam; Morgan McGuire; David F. Sklar; James D. Foley; Steven K. Feiner; Kurt Akeley, [Computer Graphics: Principles and Practice, 3rd Edition](#), *chapter 26, 27, 29*

## ◆ Papers

- ◆ Kajiya, J. T. (1986). [The Rendering Equation](#). Proc SIGGRAPH '86 Computer Graphics, 20(4), 143–150.
- ◆ Immel, D. S., Cohen, M. F., & Greenberg, D. P. (1986). [A Radiosity Method for Non-Diffuse Environments](#). Proc. SIGGRAPH '86, Computer Graphics, 20(4), 133–142.
- ◆ Eric P. Lafortune and Yves D. Willems, [Bi-Directional Path Tracing](#), Proceedings of Computergraphics '93, Alvor, Portugal (December '93), pp. 145-153.