

CG3

Introduction to Physically Based Simulation

Introduction to Physically Based Simulation

- ◆ Harmonic Oscillator
- ◆ [Physical Quantities](#)
- ◆ [Applied Analysis](#)
- ◆ [Minimization Principle](#)
- ◆ [Symmetries and Conserved Quantities](#)



Introduction to Physically Based Simulation

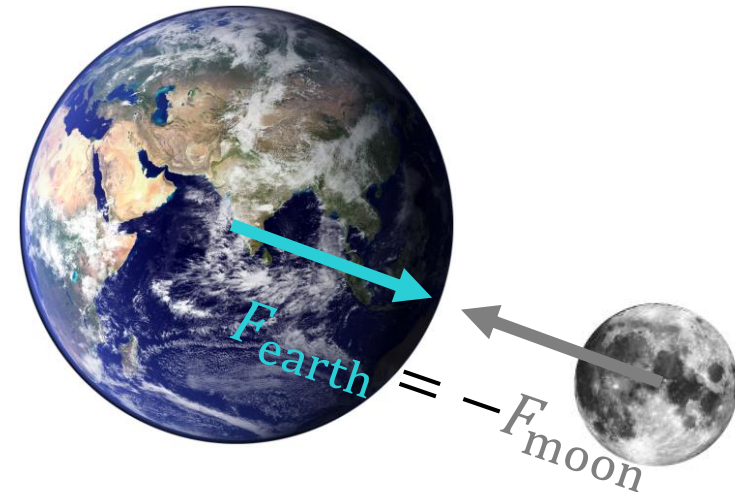
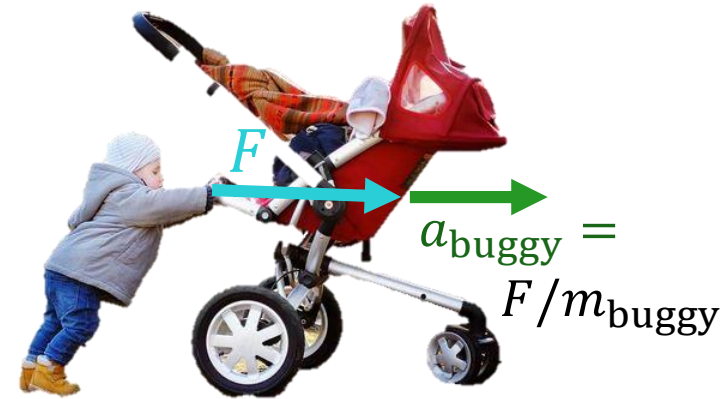
HARMONIC OSCILLATOR

Preliminaries

Newton's laws of motion



- ◆ **First law:** In an **inertial frame** of reference, an object either remains at rest or continues to move at a **constant velocity**, **unless** acted upon by a force.
- ◆ **Second law:** In an inertial frame of reference, the **vector sum** of the **forces** F on an object with constant **mass** m is equal to m multiplied by the **acceleration** a of the object:
 $F = ma$.
- ◆ **Third law:** When one body exerts a force on a second body, the second body simultaneously exerts a force **equal in magnitude** and **opposite in direction** on the first body.



Harmonic Oscillator

Analytic Solution



- Hook's law: $F_H = -kx$ (spring force F_H , stiffness constant of spring k)
- 2nd Newton's law: $F = F_H = ma = m\ddot{x}$, with $\ddot{x} = \frac{d^2x}{dt^2}$
- eq. of motion: $m\ddot{x} = -kx$, with $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$
- solution to initial value problem:

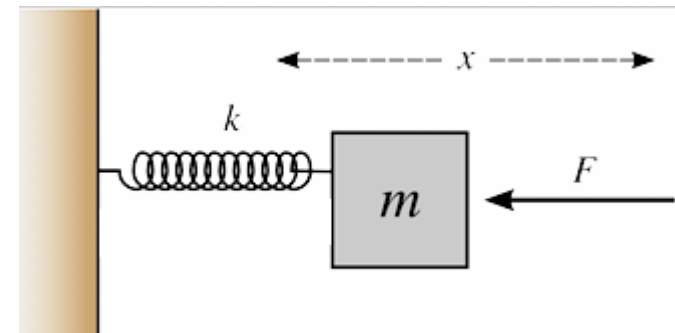
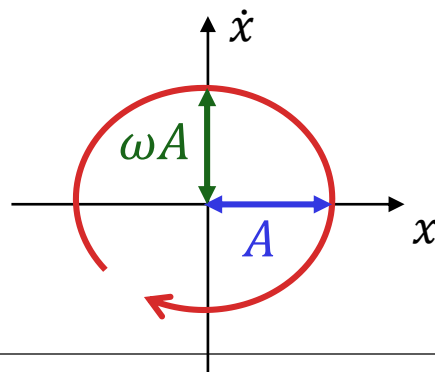
$$x(t) = A \cdot \cos(\omega t + \phi) \quad \text{with } \omega = \sqrt{\frac{k}{m}}, \quad [A, \phi](x_0, \dot{x}_0)$$

(angular frequency ω , phase ϕ)

$$\dot{x}(t) = -\omega A \cdot \sin(\omega t + \phi)$$

$$\ddot{x}(t) = -\omega^2 A \cdot \cos(\omega t + \phi)$$

- state space diagram:

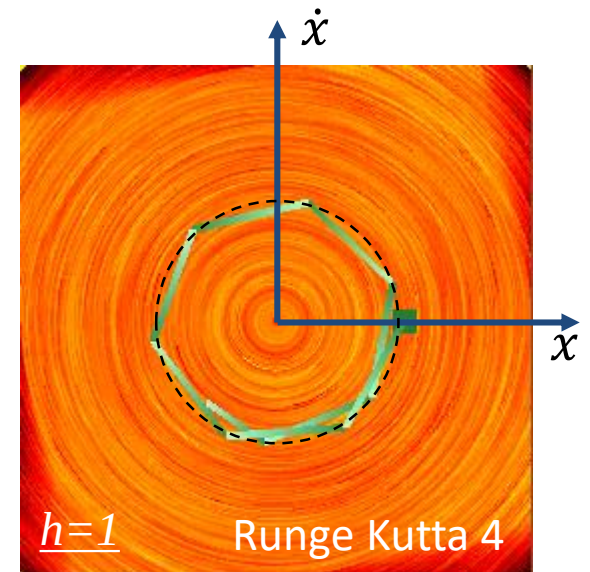
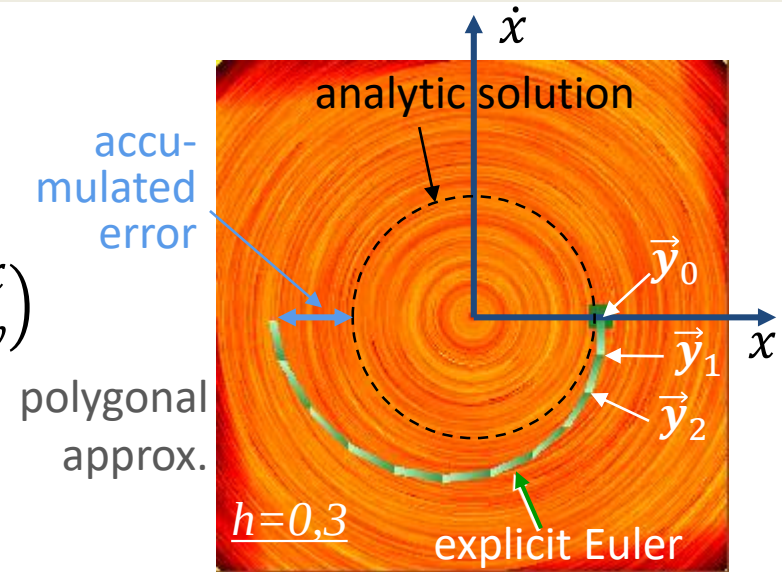


Harmonic Oscillator

Numeric Solution



- combine location and velocity to state vector $\vec{y} = \begin{pmatrix} x \\ v \equiv \dot{x} \end{pmatrix}$
- rewrite equations of motion
$$m\ddot{x} = -kx \Rightarrow \dot{\vec{y}}_i = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$
- start with initial state $\vec{y}_0 = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$
- discretize state space trajectory with polygon through $\vec{y}_{i=0\dots n}$
- simplest approach is explicit Euler:
$$\vec{y}_{i+1} = \vec{y}_i + h\dot{\vec{y}}_i$$
with step width h .
- very small step widths necessary for explicit Euler
- use higher order methods





Introduction to Physically Based Simulation

PHYSICAL QUANTITIES



- A **physical constant** is a physical quantity that is generally believed to be both universal in nature and constant in time.
- The concrete value of a physical constant depends on the chosen units.
- In SI units the most fundamental constants are
 - speed of light in vacuum c 299 792 458 $\text{m}\cdot\text{s}^{-1}$
 - Newtonian constant of gravitation γ 6.674 30(15) $\times 10^{-11} \text{m}^3\cdot\text{kg}^{-1}\cdot\text{s}^{-2}$
 - Planck constant h 6.626 070 15 $\times 10^{-34} \text{J}\cdot\text{s}$
- The appearance of our universe strongly depends on the values of these constants
- **Anthropic principle**: the values of the constants must be like that as we do observe them as living beings.

uncertainty in
last two digits

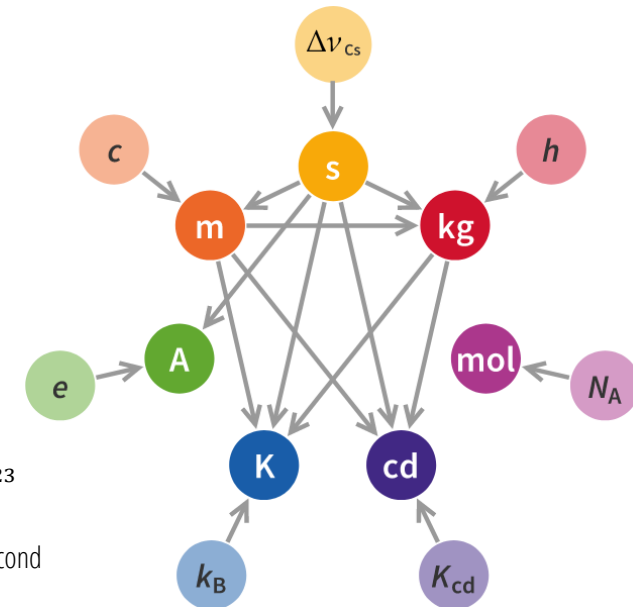
Physical Quantities

Physical Units

- A physical quantity is always accompanied by a physical unit
- **Système International d'unités** defines the 7 base units
 - [Kelvin](#) (temperature), ^{1/273.16 of temperature of the triple point of water}
 - [second](#) (time), ^{9192631770 cycles of a Caesium [atomic clock](#)}
 - [meter](#) (distance), ^{path length travelled by light in a vacuum in 1/299792458 second}
 - [kilogram](#) (mass), <sup>mass of [Big K](#), till Nov 2018,
now from $h = 6.62607015 \cdot 10^{-34} \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$</sup>
 - [candela](#) (luminous intensity),
 - [mole](#) (amount of substance) <sup>number $N_A = 6.02214076 \times 10^{23}$
of atoms in 12g ^{12}C</sup>
 - [Ampere](#) (electric current) <sup>electric current carried by $1/e$ electrons per second
with $e = 1.602176634 \cdot 10^{-19}\text{C}$</sup>
- derived units can be written with a scale n and 7 exponents in terms of a base unit:

$$10^n \cdot \text{m}^\alpha \cdot \text{kg}^\beta \cdot \text{s}^\gamma \cdot \text{A}^\delta \cdot \text{K}^\varepsilon \cdot \text{mol}^\zeta \cdot \text{cd}^\eta$$

$$l = 1\text{m}, v = 5\frac{\text{m}}{\text{s}}, m = 2\text{kg}, \dots$$



Dependence of 7 SI base unit definitions on physical constants with fixed numerical values and on other base units

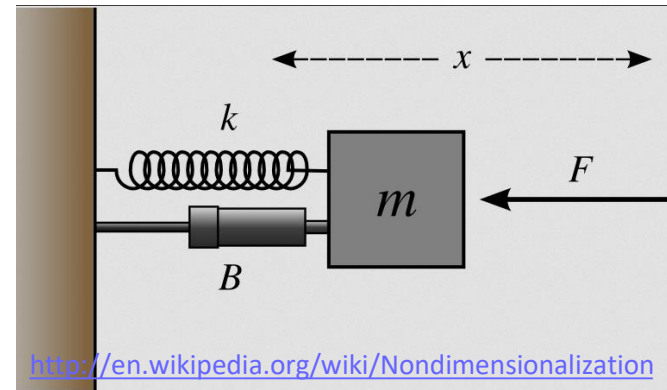
$$\text{Ohm: } \Omega = \text{m}^2 \cdot \text{kg}^1 \cdot \text{s}^{-3} \cdot \text{A}^{-2}$$

- Physical units help to validate physical formulas and to derive units of physical constants:
 - one cannot **add** quantities of different units
 - left and right side of an **equality** must have the same unit
 - The arguments of functions like **sin**, **cos**, **exp**, ... cannot have a unit (also steradian is not allowed here)

Example: damped harmonic oscillator

- $m \frac{d^2 x}{dt^2} + B \frac{dx}{dt} + kx = F(t)$
- What unit has damping constant?

$$[B] = [F(t)] / \left[\frac{dx}{dt} \right]$$
$$= \text{kg} \cdot \text{m} \cdot \text{s}^{-2} / \text{m} \cdot \text{s}^{-1} = \text{kg} \cdot \text{s}^{-1}$$



x = displacement from equilibrium [m]

t = time [s]

F = external force [$\text{kg} \cdot \text{m} \cdot \text{s}^{-2}$]

m = mass of the block [kg]

B = damping constant

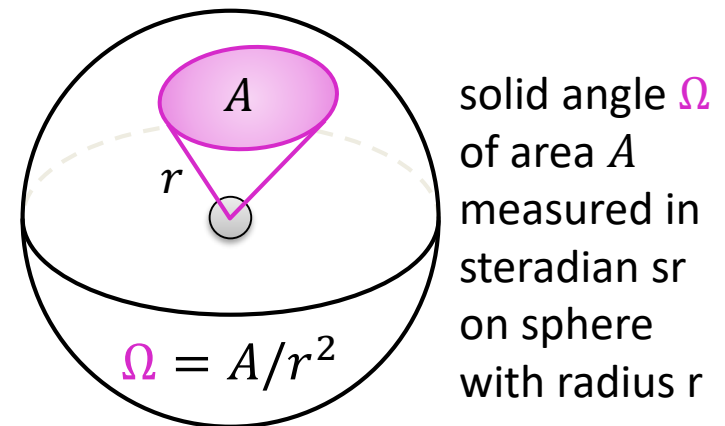
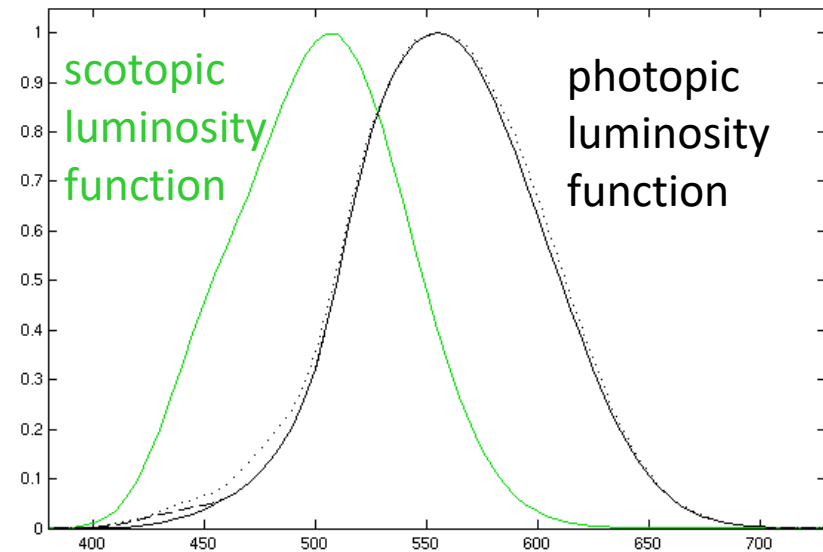
k = force constant of spring [$\text{kg} \cdot \text{s}^{-2}$]

Physical Quantities

Candela and Luminous Intensity



- origin: 1 **candela** measures how bright the human eye perceives a wax candle.
- quantity: **luminous intensity**
- Brightness perception of human eye increases with light power and depends on wavelength through the **CIE** standardized **photopic luminosity function** $\bar{y}(\lambda)$
- To relate light intensity (emitting from a point) to power [Watt] one integrates intensity over direction
- **Solid angle** measures directions by their covered area on the unit sphere and is measured in **steradian (sr)**

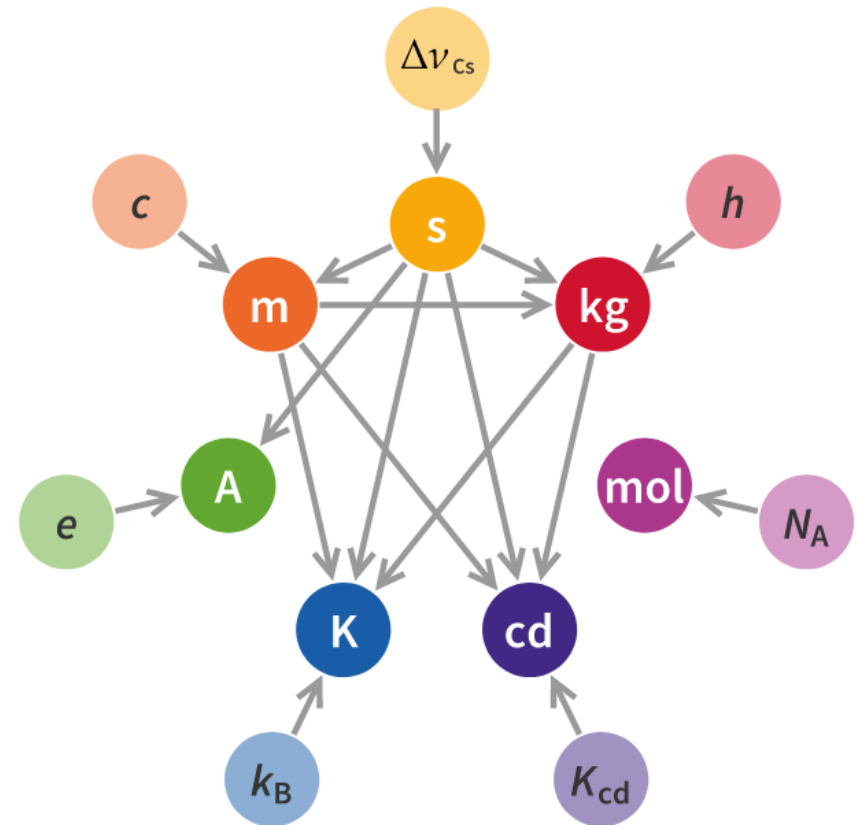


Physical Quantities

Candela and Luminous Intensity



- ◆ **Def.:** 1 **candela** in given direction is luminous intensity of light source emitting $1/683$ Watt of monochromatic green ($K_{cd} = 540 \times 10^{12} \text{ Hz}$) light per sr.
- ◆ Its definition depends on Watt, Hertz and steradian and therefore on m, s, and kg:
 - ◆ hertz: $\text{Hz} = \text{s}^{-1}$
 - ◆ watt: $\text{W} = \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3}$
 - ◆ steradian: $\text{sr} = \text{m}^2 \cdot \text{m}^{-2}$
- ◆ For steradian all base units cancel out, but we still write it as sr to distinguish from a pure scalar.



Physical Quantities

Standard and Non-Standard Units



For physically based graphics (excluding molecular and electro dynamics) the following standard and non-standard units are important

Second	s	duration, time	s
Meter	m	length, position, size	m
Kilogram	kg	mass	kg
Kelvin	K	temperature	K
Candela	cd	luminous intensity	cd
Steradian	sr	solid angle	$m^2 \cdot m^{-2}$
Lumen	lm	luminous flux	cd·sr
Lux	lx	illuminance	$lm/m^2 = cd \cdot sr \cdot m^{-2}$
Hertz	Hz	frequency	s^{-1}
Newton	N	force	$kg \cdot m \cdot s^{-2}$
Pascal	Pa	pressure	$N/m^2 = kg \cdot m^{-1} \cdot s^{-2}$
Joule	J	energy, work, heat	$N \cdot m = kg \cdot m^2 \cdot s^{-2}$
Watt	W	power, radiant flux	$J/s = kg \cdot m^2 \cdot s^{-3}$

Physical Quantities

Nondimensionalization

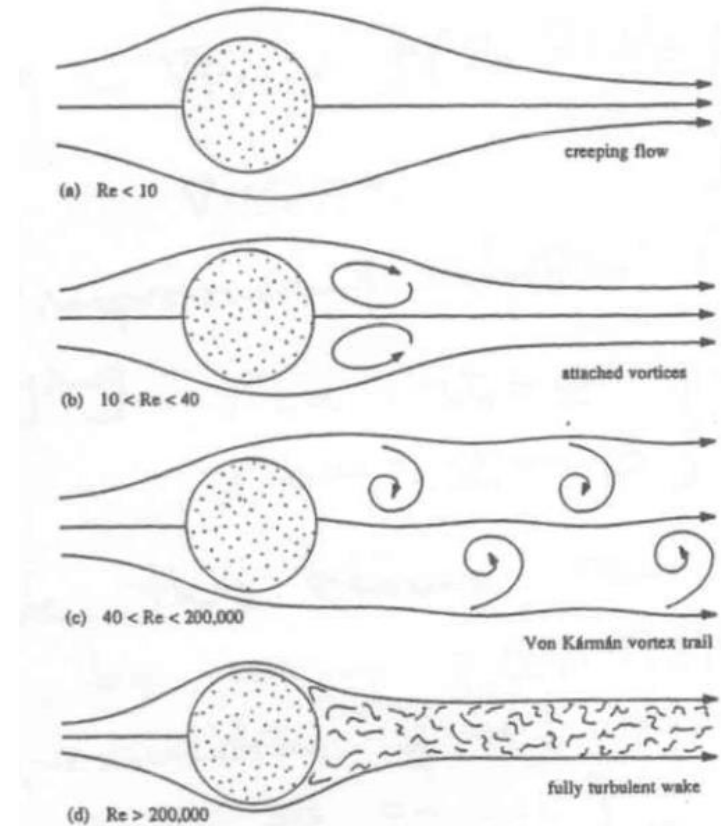


- ◆ If the physical laws of a system are known one can get rid of all units by a simple variable substitution.
- ◆ The remaining constants are dimensionless and parameterize different behaviors of the system.

Example:

- ◆ dimensionless Navier Stokes Equations of incompressible fluids are parametrized over Reynolds number Re :

$$\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \sum_j \frac{\partial^2 u_i}{\partial x_j^2} + f \Big|_{i \in \{1,2,3\}} \quad \text{and} \quad \sum_j \frac{\partial u_j}{\partial x_j} = 0$$



Patterns in fluid flow around a cylinder as a function of the Reynolds number.

Image Source: <http://www.hitech-projects.com/euprojects/artic/index/Low%20Reynolds%20number%20flows.pdf>



Example: damped harmonic oscillator

$$m \frac{d^2 x}{dt^2} + B \frac{dx}{dt} + kx = AF(t)$$

- ◆ Substitutions: $\tau = \frac{t}{t_c}$ and $\chi = \frac{x}{x_c}$

$$m \frac{x_c}{t_c^2} \frac{d^2 \chi}{d\tau^2} + B \frac{x_c}{t_c} \frac{d\chi}{d\tau} + k x_c \chi = AF(\tau t_c)$$

- ◆ Division of constant from highest derivative order term:

$$\frac{d^2 \chi}{d\tau^2} + t_c \frac{B}{m} \frac{d\chi}{d\tau} + t_c^2 \frac{k}{m} \chi = \frac{t_c^2}{m x_c} Af(\tau)$$

- ◆ Define constant $t_c = \sqrt{\frac{m}{k}}$ and $x_c = \frac{A}{k}$

$$\frac{d^2 \chi}{d\tau^2} + 2\zeta \frac{d\chi}{d\tau} + \chi = f(\tau), \text{ with damping ratio } 2\zeta = \frac{B}{\sqrt{mk}}$$

- ◆ unit free damping ratio defines system behavior



- ◆ Physical quantities carry units that help to validate formula and to interpret constants
- ◆ SI defines 7 base units and expresses all other units as powers of these
- ◆ One of these 7 is Candela, which is used for photogrammetric measurements of visible light intensity relative to the spectral sensitivity of the human eye.
- ◆ A few physical constants define the physics in our universe that would look very different if their values would change slightly
- ◆ For a given physical system nondimensionalization allows to describe the system with unit free variables and a few characteristic unit free parameters





Introduction to Physically Based Simulation

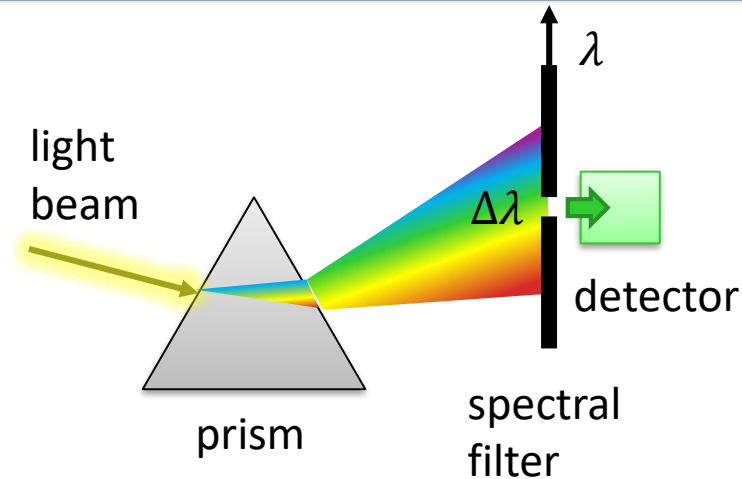
APPLIED ANALYSIS

Applied Analysis

One Independent Variable



- The **specialization** of a quantity with respect to some independent variable is mathematically describes as a derivative
- From a physical point of view one designs a **filter** that restricts the measurement process to a small interval of the independent variable
- We need specialization with respect to time to learn about the time evolution
 - power is work done per time
 - velocity is path length change per time
 - acceleration is velocity change per time
- Other important independent variables are location, direction and wavelength
 - radiant power per wavelength is **spectral power**



$$\text{power: } P = \frac{dW}{dt}$$

$$\text{velocity: } v = \frac{ds}{dt}$$

$$\text{acceleration: } a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

$$\text{spectral power: } \Phi_\lambda = \frac{d\Phi}{d\lambda}$$



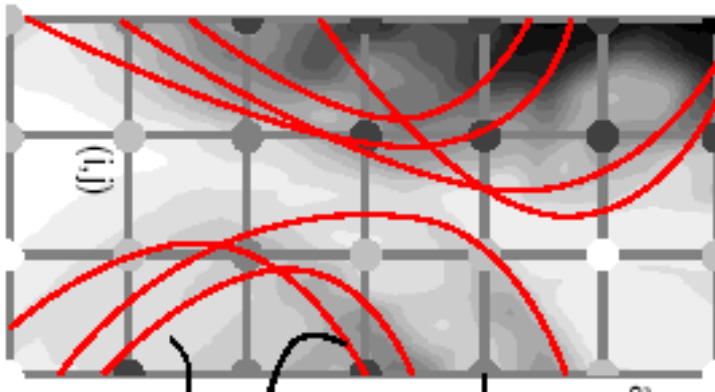
- ◆ triple derivative of mass m
with respect to volume V
yields density ρ
- ◆ unit of density is kg/m^3
- ◆ triple integration of density
gives back mass

$$\rho = \frac{dm}{dV} = \frac{d^3 m(x, y, z)}{dx dy dz}$$

$$m = \iiint_V \rho(x, y, z) dx dy dz$$

Eulerian View

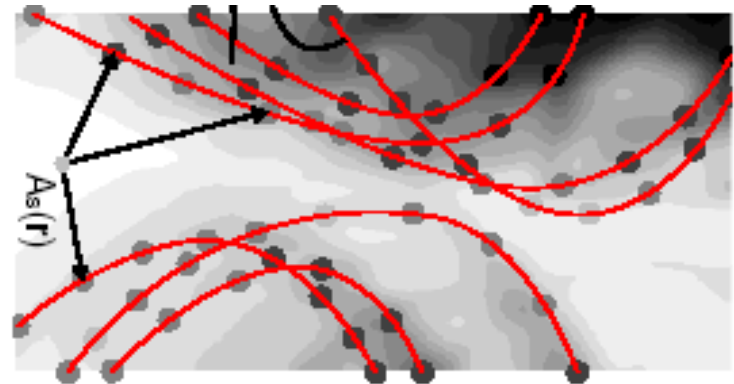
- Describe physics as fields (flow field, irradiance fields, ...) over space
- For simulation fields are often discretized over grids or meshes and finite difference or finite element methods are applied.



fields are easy to implement

Lagrangian View

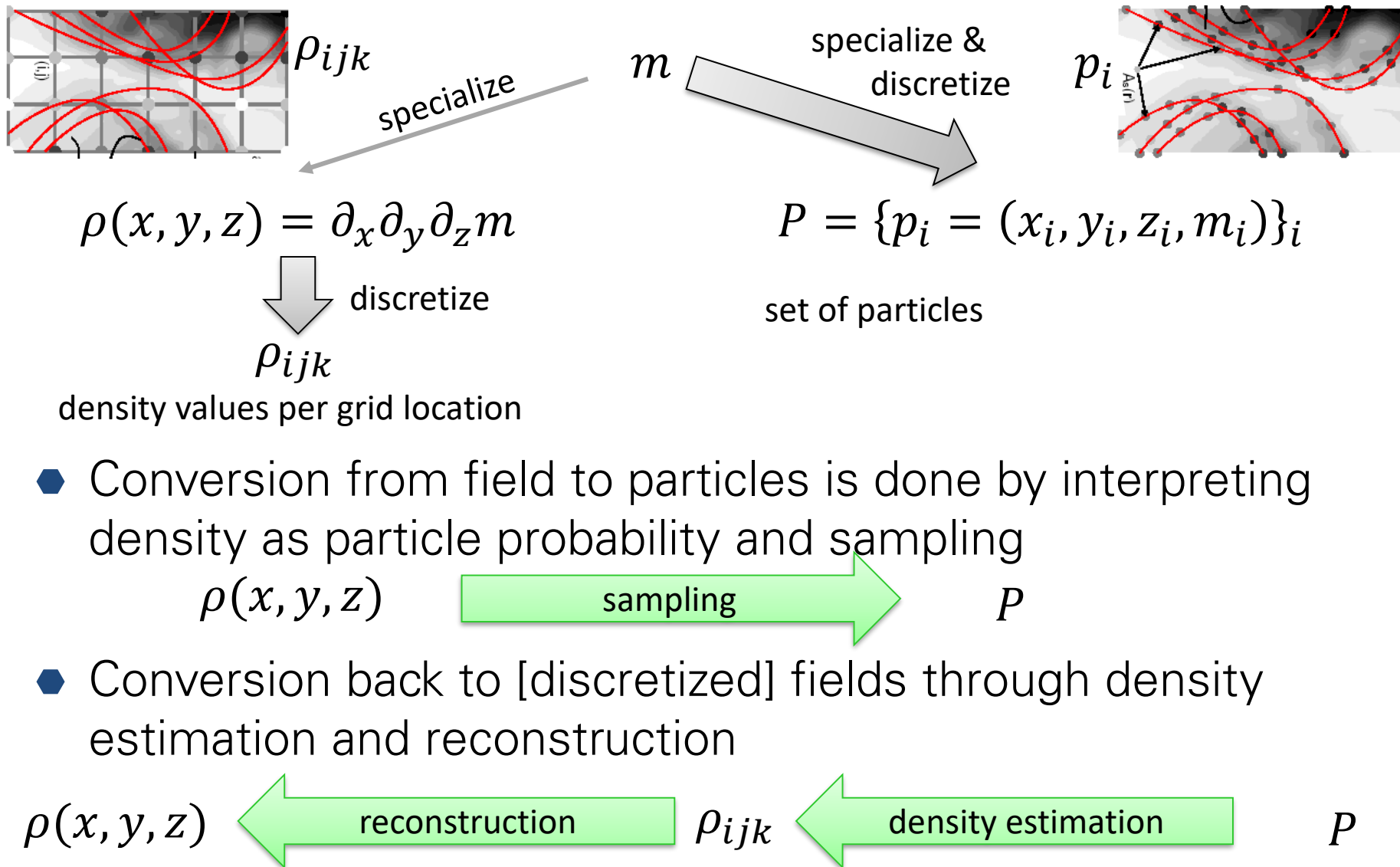
- Describe physics in form of particles that move in space
- During particle simulations, particles are the discretization unit and typically do not represent single physical particles (photons, molecules, ...) but bundles of them



particles are conceptually simple

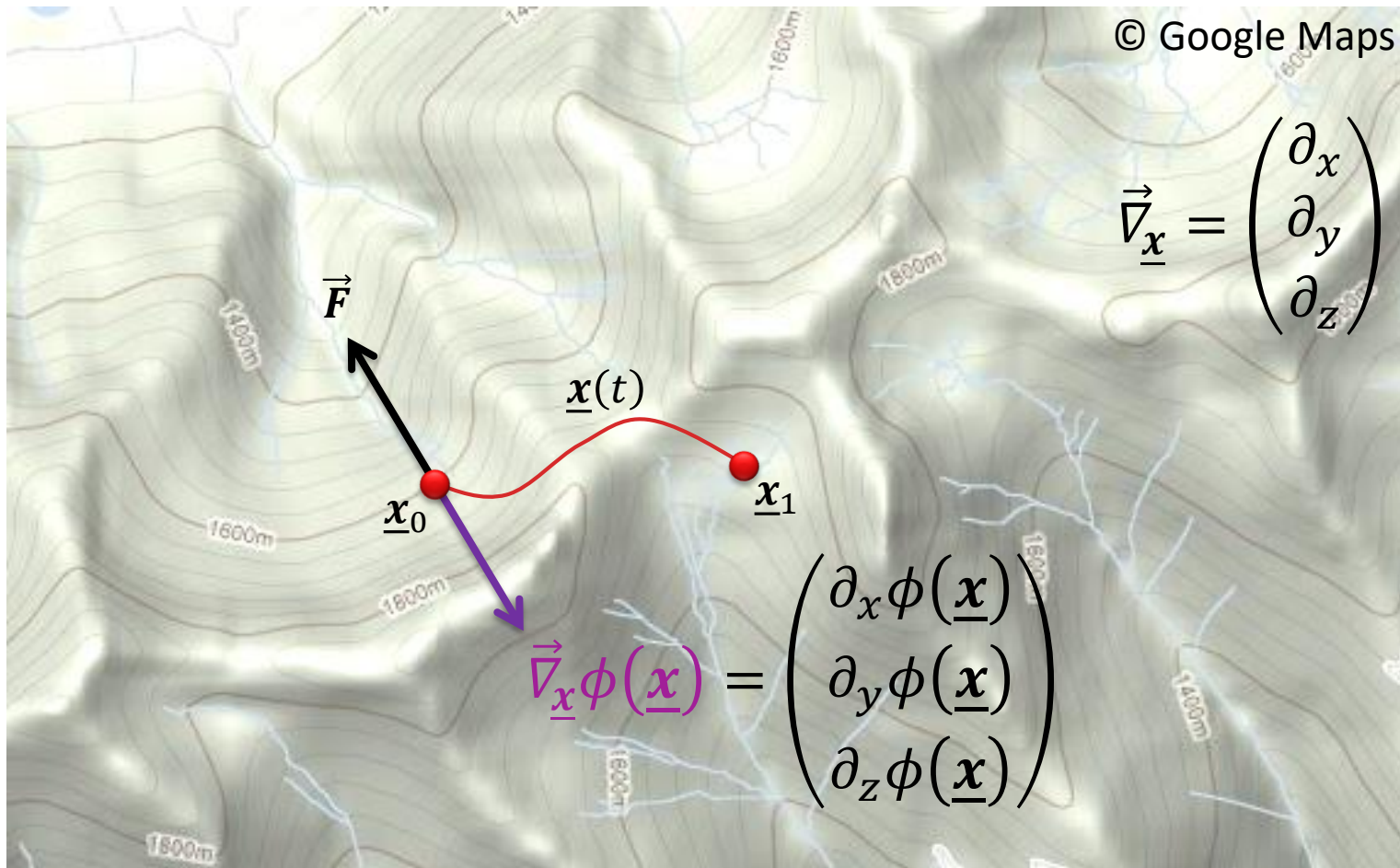
Applied Analysis

Field to Particles and Back



Applied Analysis

Conservative Force, Potential Energy



gradient is direction of steepest ascent, force points in opposite direction

Example: Potential energy for near-Earth gravity $\phi_g(\underline{x}) = m \cdot g \cdot h(\underline{x})$



- ◆ negated gradient of potential energy ϕ with respect to location $\underline{x}$ yields **conservative force** $\vec{F}$. Unit: $N = J/m$
- ◆ work W done by force from integration along path $\underline{x}(t)$
- ◆ **gradient theorem** states that work done by force is potential difference of path end points
- ◆ no work done nor necessary for cyclic paths where $\underline{x}_0 = \underline{x}_1$
- ◆ force is **conservative**, iff curl of force vanishes everywhere

$$\vec{F} = -\vec{\nabla}_{\underline{x}}\phi(\underline{x})$$

$$W = \int_{\underline{x}_0}^{\underline{x}_1} \vec{F} d\underline{x} = \int_{t_0}^{t_1} \langle \vec{F}, \vec{\dot{x}} \rangle dt$$

$$\begin{aligned} W &= - \int_{\underline{x}_0}^{\underline{x}_1} \vec{\nabla}_{\underline{x}}\phi(\underline{x}) d\underline{x} \\ &= -\phi(\underline{x}) \Big|_{\underline{x}_0}^{\underline{x}_1} = \phi(\underline{x}_0) - \phi(\underline{x}_1) \end{aligned}$$

$$\vec{\nabla}_{\underline{x}} \times \vec{F} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix} = \vec{0}$$

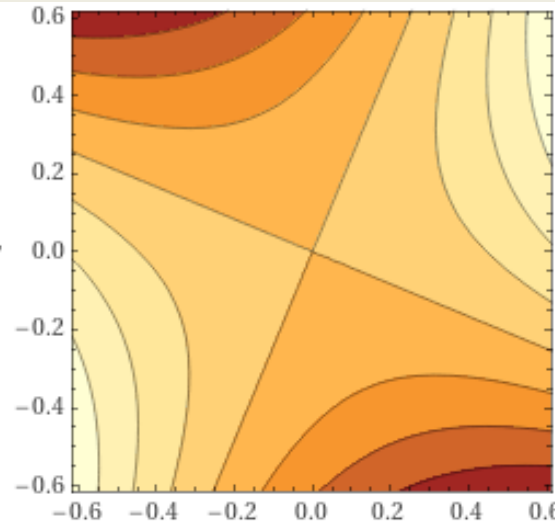
Example 1 – Conservative Force

$$\phi(\underline{x}) = x^2 + 2xy - y^2$$

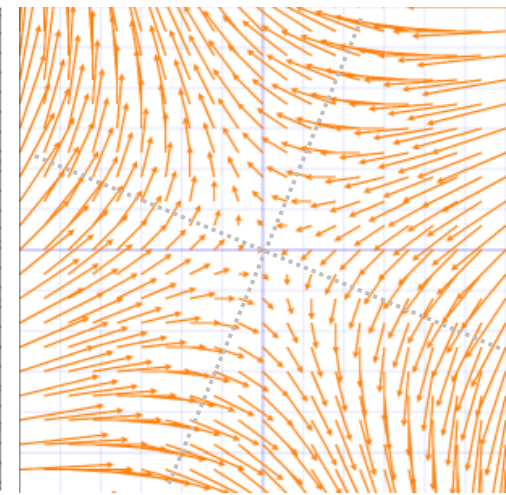
$$F_x = -\partial_x \phi = -2x - 2y$$

$$F_y = -\partial_y \phi = -2x + 2y$$

$$\begin{aligned} \vec{\nabla} \times \vec{F} \Big|_z &= \partial_x F_y - \partial_y F_x \\ &= -2 - (-2) = 0 \end{aligned}$$



$$\phi(\underline{x})$$



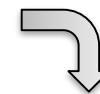
$$\vec{F} = -2 \begin{pmatrix} x + y \\ x - y \end{pmatrix}$$

$$-\int F_x dx = x^2 + xy + C_x$$

$\parallel$



$$\begin{aligned} C_x &= -y^2 + C \\ C_y &= x^2 + C \end{aligned}$$



$$-\int F_y dy = -y^2 + xy + C_y$$

$$\phi(\underline{x}) = x^2 + 2xy - y^2 + C$$

Example 2 – Non Conservative Force



$$\vec{F} = 2 \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\begin{aligned} \vec{\nabla} \times \vec{F} \Big|_z &= \partial_x F_y - \partial_y F_x \\ &= 2 - (-2) = 4 \neq 0 \end{aligned}$$

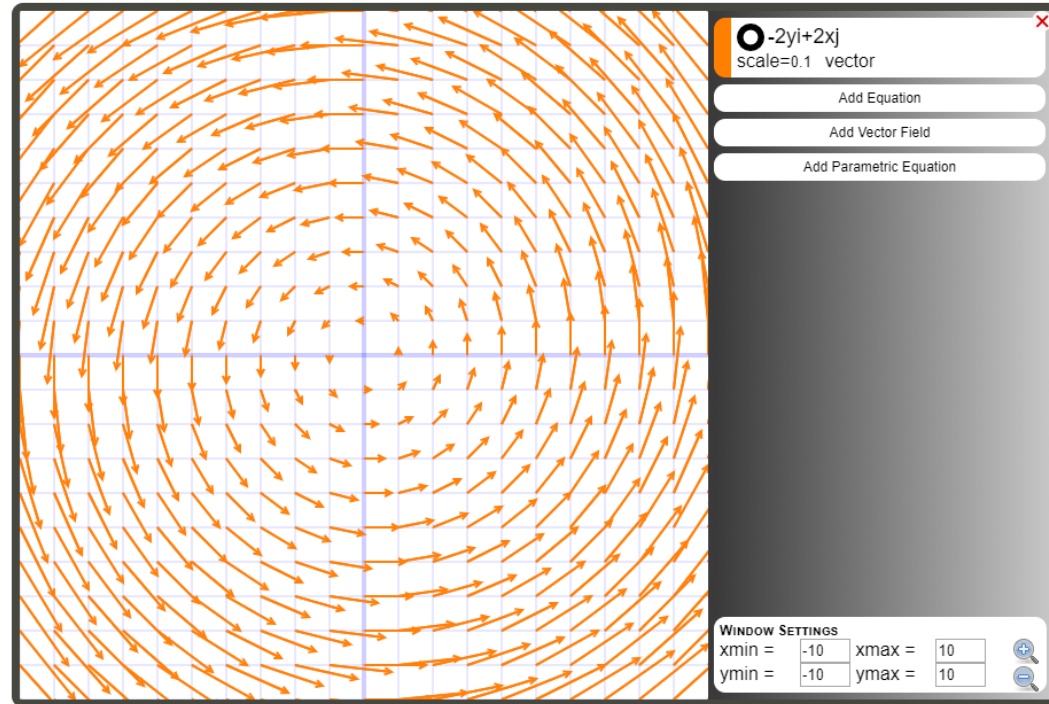
$$- \int F_x dx = 2xy + C_x$$

$\parallel \Rightarrow$

$$- \int F_y dy = -2xy + C_y$$

$$C_x = ?$$

$$C_y = ?$$

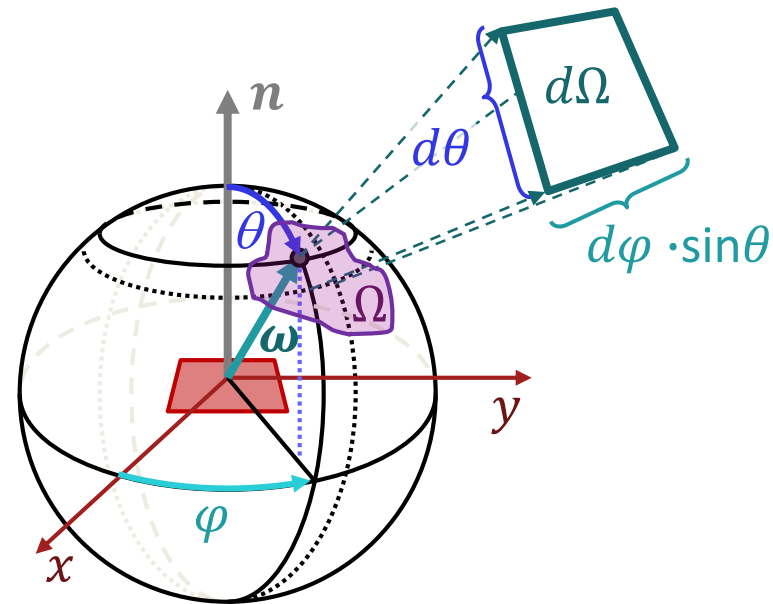


Applied Analysis

Directions and Solid Angle



- Solid angle is used to measure a set of directions ω represented as unit vectors
- Typically ω is parameterized over the *unit sphere* in spherical coordinates φ, θ
- *This parametrization is relative to local surface normal $\mathbf{n}$*
- **solid angle Ω** is measured in area covered on unit sphere
- nonstandard unit: *sr (steradian)*
- integration of solid angle yields double integral over φ and θ



$$\omega = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}, \quad \begin{matrix} \varphi \in [-\pi, \pi] \\ \theta \in [0, \pi] \end{matrix}$$

$$d\Omega = d\varphi \cdot \sin\theta d\theta$$

Applied Analysis

Directions and Solid Angle



- ◆ The solid angle corresponding to **all directions** is 4π
- ◆ The solid angle of a **hemisphere** (directions to the outside at surface point), is therefore 2π
- ◆ One just needs to integrate 1 over spherical coordinates to show that:

$$\Omega^{\text{all}} = \iint_{\Omega^{\text{all}}} 1 \, d\Omega = \int_0^\pi \int_{-\pi}^\pi 1 \, d\varphi \sin\theta \, d\theta = \int_0^\pi \left(\int_{-\pi}^\pi 1 \, d\varphi \right) \cdot \sin\theta \, d\theta$$

$$= 2\pi \int_0^\pi \sin\theta \, d\theta = 2\pi [-\cos\theta]_0^\pi = 2\pi(1 - (-1)) = 4\pi$$



- ◆ a lot of physical quantities are derivatives of others
- ◆ the variables with respect to which the derivation is applied, add their units to the denominator
- ◆ with respect to location one can do
 - ◆ triple derivatives yielding again a scalar density or
 - ◆ gradients that result in vector valued quantities like forces
- ◆ integration of conservative forces along paths can be computed from differences in potential energy
- ◆ not all force fields can be integrated into potential, only the ones where the curl vector vanishes everywhere
- ◆ when integrating directions in spherical coordinates an additional $\sin\theta$ is needed





Introduction to Physically Based Simulation

MINIMIZATION PRINCIPLE

Minimization Principle

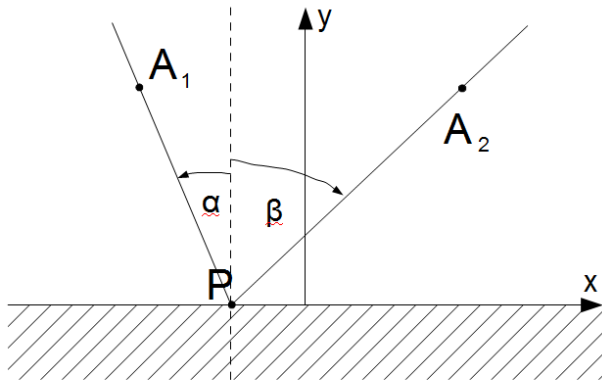
Fermat's Principle



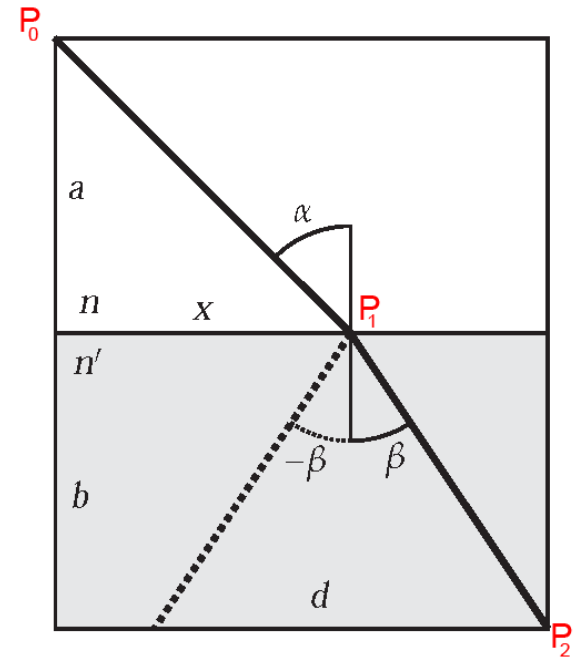
Light travels along the shortest path with respect to time

- from this the laws for reflection and refraction follow

http://de.wikipedia.org/wiki/Fermatsches_Prinzip



$$\alpha = \beta$$



$$\frac{c_2}{c_1} = \frac{\sin \beta}{\sin \alpha}$$



- mechanical systems can be completely described through the scalar **Lagrangian** L that depends on the time dependent state vector $\vec{y}(t)$ of the system and potentially on time t :

$$L(\vec{y}(t), t) = T - V$$

- with the **kinetic energy** T and the **potential energy** V .
- the **state vector** $\vec{y}$ contains all object positions and velocities
- the **action** S of the system is defined as the functional that maps the time evolution of the system state to a scalar:

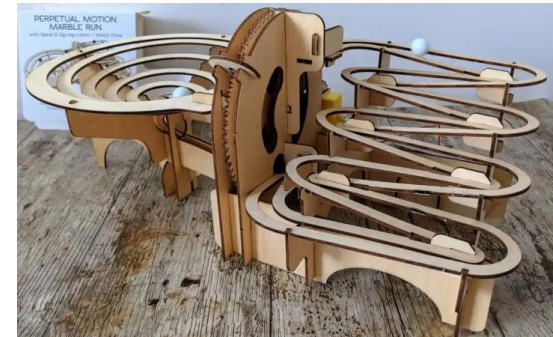
$$S[L](t_1, t_2) = \int_{t_1}^{t_2} L(\vec{y}(t), t) dt$$

*The path $\vec{y}(t)$ taken by the system between times t_1 and t_2 is the one for which the **action is stationary** (no change) to first order.*



- With variational calculus the **Euler Lagrange Equations** can be derived from the principle of stationary action:

$$\forall i: \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$



- here i enumerates the generalized positions q_i and generalized velocities $\dot{q}_i$.

Example: harmonic oscillator

- $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

- $\frac{\partial L}{\partial \dot{x}} = m\dot{x} , \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = m\ddot{x} , \quad \frac{\partial L}{\partial x} = -kx \Rightarrow m\ddot{x} = -kx$

Minimization Principle

Summary



- ◆ the dynamics of physical systems can be formulated as minimization problem
- ◆ examples:
 - ◆ Fermat's Principle (shortest paths)
 - ◆ Least Action Principle
- ◆ if minimization is over functions, one needs variational calculus
- ◆ from Least Action Principle one can derive the Euler Lagrange Equations that generalize equations of motions





Introduction to Physically Based Simulation

SYMMETRIES AND CONSERVED QUANTITIES



Any differentiable symmetry of the action of a physical system has a corresponding conservation law.

Examples

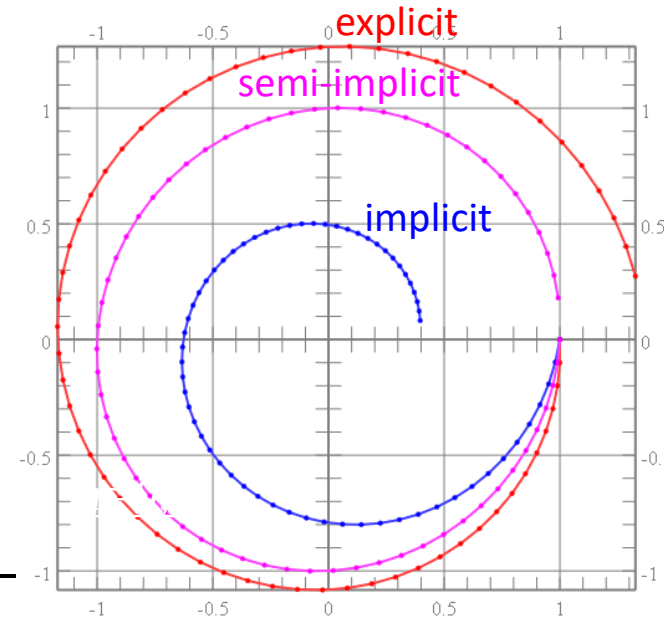
- ◆ **time** symmetry: As laws of physics / experiments do not depend on when they are done, **energy** is conserved
- ◆ **location** symmetry: As laws of physics / experiments do not depend on where they are done, **linear momentum** (**mass times linear velocity**) is conserved
- ◆ **orientation** symmetry: As laws of physics / experiments do not depend on their spatial orientation, **angular momentum** (**inertia tensor times angular velocity**) is conserved

Symmetries & Conserved Quantities

Symplectic Numerical Integration



- Explicit** numerical integration techniques like the explicit Euler add energy to the system (system becomes instable)
- Implicit** integration techniques are stable but unnaturally damp the system and remove energy
- Symplectic** integrators conserve energy as good as possible but are not stable for stiff systems.



$\dot{x} = f(v, t)$
 $\dot{v} = g(x, t)$
 system of
 diff. equa.

$x_{i+1} = x_i + h \cdot f(v_i, t)$
 $v_{i+1} = v_i + h \cdot g(x_i, t)$
 explicit Euler

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 $v_{i+1} = v_i + h \cdot g(x_{i+1}, t)$
 implicit Euler

$v_{i+1} = v_i + h \cdot g(x_i, t)$
 $x_{i+1} = x_i + h \cdot f(v_{i+1}, t)$
 semi-implicit /
 symplectic Euler

Symmetries & Conserved Quantities

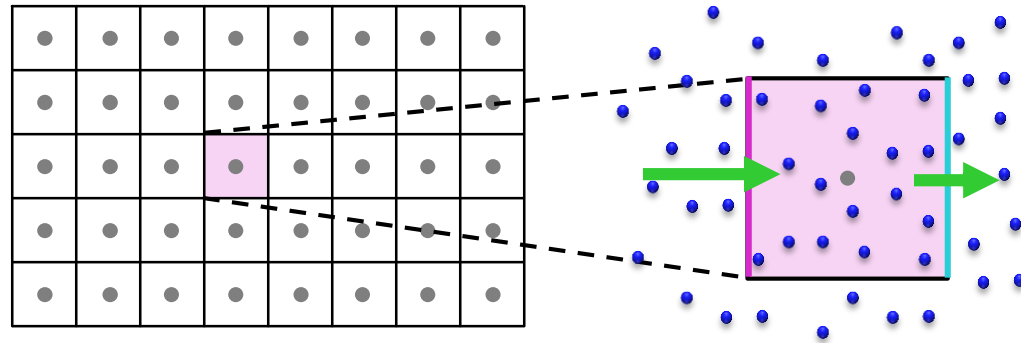
Continuity equation



Example: mass preservation

Lagrangian view

- each particle p_i carries mass m_i ; automatic preservation if particles persist



Eulerian view

- mass is represented as density field $\rho(\underline{x})$ over grid
- cell volume: $dV = dx^3$
- imagine mass is split into equal sized particles of density $\pi(\underline{x})$ and mass m
- mass density: $\rho = m \cdot \underbrace{\pi \cdot dV}_{\substack{\text{nr. part.} \\ \text{in cell}}} / dV$
- each particle travels with velocity $\vec{u}(\underline{x})$

How does particle movement change ρ over time step dt ?

- Let's restrict motion to x -direction u_1 of $\vec{u}$ & examine one cell
- traveled distance: $u_1 \cdot dt$
- particle in/outflow(left/right):
 $dx^2 \cdot (\pi \cdot u_1)(x \mp dx/2) \cdot dt$
- change in mass density:
$$\frac{\partial \rho}{\partial t} = -m \cdot \frac{\partial(\pi \cdot u_1)}{\partial x} = -\frac{\partial(\rho \cdot u_1)}{\partial x}$$

Symmetries & Conserved Quantities

Continuity equation



- as we have the same in/outflows along the other coordinate directions, we get:

$$\frac{\partial \rho}{\partial t}(\underline{x}) + \frac{\partial (\rho(\underline{x}) \cdot u_1(\underline{x}))}{\partial x} + \frac{\partial (\rho(\underline{x}) \cdot u_2(\underline{x}))}{\partial y} + \frac{\partial (\rho(\underline{x}) \cdot u_3(\underline{x}))}{\partial z} = 0$$

- introducing the **mass flux vector** $\vec{j} = \rho \cdot \vec{u}$ this simplifies:

$$\frac{\partial \rho}{\partial t}(\underline{x}) + \frac{\partial j_1(\underline{x})}{\partial x} + \frac{\partial j_2(\underline{x})}{\partial y} + \frac{\partial j_3(\underline{x})}{\partial z} = 0$$

- the formula simplifies further by introducing the **divergence operator** $\vec{\nabla} \cdot \vec{v} = \text{div } \vec{v} = \partial_x v_1 + \partial_y v_2 + \partial_z v_3$:

$$\partial_t \rho + \text{div}(\vec{j}) = \partial_t \rho + \text{div}(\rho \vec{u}) = 0$$

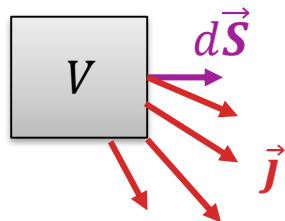
- Finally we introduce a **volumetric mass source** $\sigma(\underline{x})$ yielding the mass continuity equation:

$$\partial_t \rho + \text{div}(\rho \vec{u}) = \sigma$$

Symmetries & Conserved Quantities

Continuity equation

- ◆ The continuity equation can be constructed for any other quantity $q(\underline{x})$ carried with fluid particles like electric charge by exchanging the symbol ρ with q .
- ◆ An alternative derivation from density $\rho(\underline{x})$, velocity $\vec{u}(\underline{x})$, flux $\vec{J}(\underline{x})$ and source $\sigma(\underline{x})$ fields results from an integral formulation, that keeps track on all changes in q :



$$\frac{d\rho}{dt} + \oiint_S \vec{J} \cdot d\vec{S} = \iiint_V \sigma dV$$

$d\vec{S}$...infinitesimal
surface area times
surface normal

- ◆ The differential form is given again as:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{J} = \sigma$$

- ◆ For conserved quantities $\sigma \equiv 0$.

Symmetries & Conserved Quantities

Summary



- ◆ It is important to be aware of quantities that are conserved in physical systems like energy
- ◆ These are due to spatial and temporal symmetries in the laws of physics
- ◆ Symplectic numerical integration methods target for energy preservation
- ◆ Continuity equations describe temporal changes of physical quantities inside of fluids or fields. For conserved quantities they do not have a source term.

