

The Rendering Equation

A Top-Down Approach

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Overview

The Rendering Equation

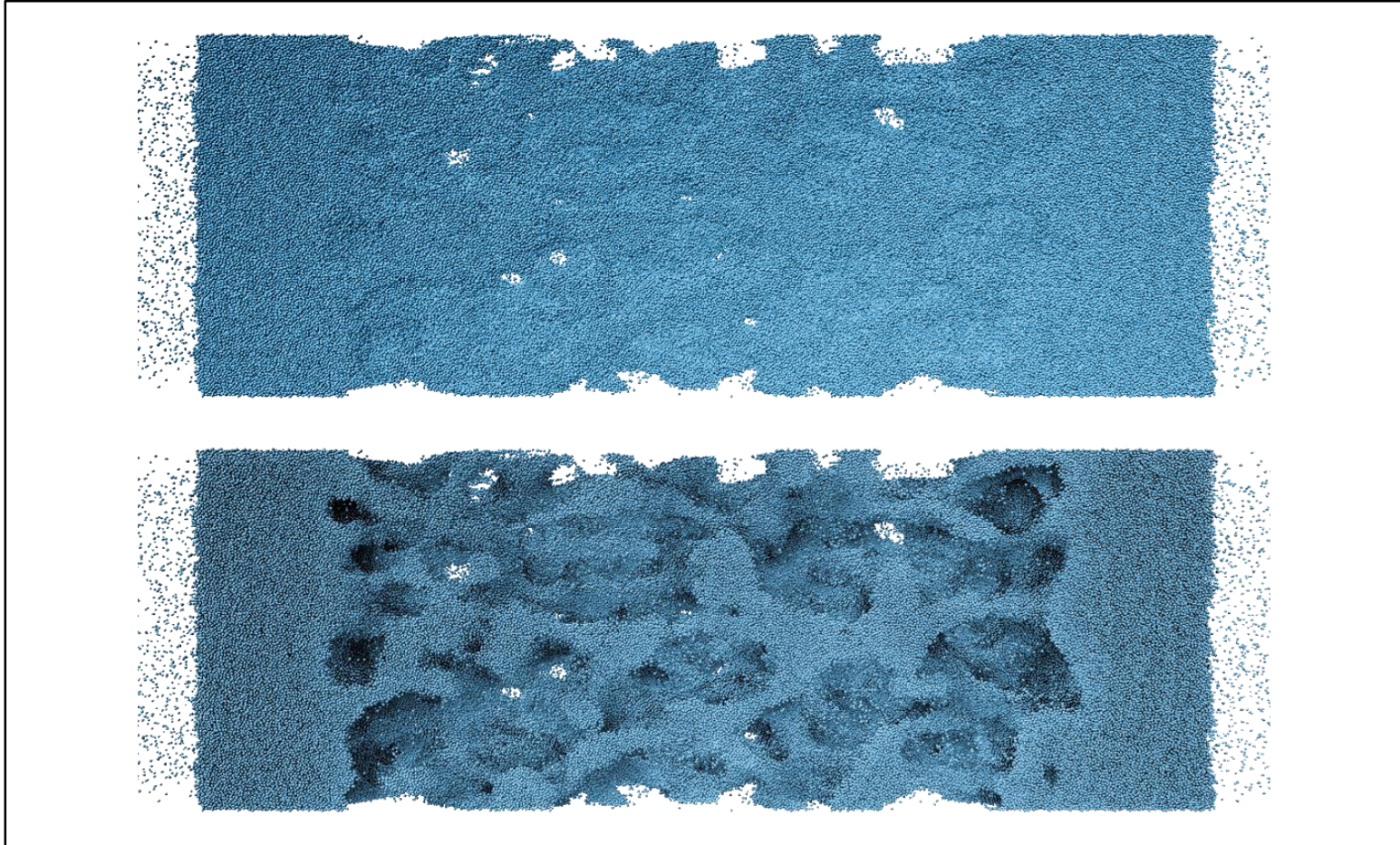
- ◆ motivation
- ◆ light transport and rendering equation
- ◆ basic radiometry and hemispherical reflectance
- ◆ numerical solution strategies

Literature

- ◆ [Kajiya](#), J. T. (1986). The Rendering Equation. Proc SIGGRAPH '86 Computer Graphics, 20(4), 143–150.
- ◆ Immel, D. S., Cohen, M. F., & Greenberg, D. P. (1986). A Radiosity Method for Non-Diffuse Environments. Proc. SIGGRAPH '86, Computer Graphics, 20(4), 133–142.

Global Illumination

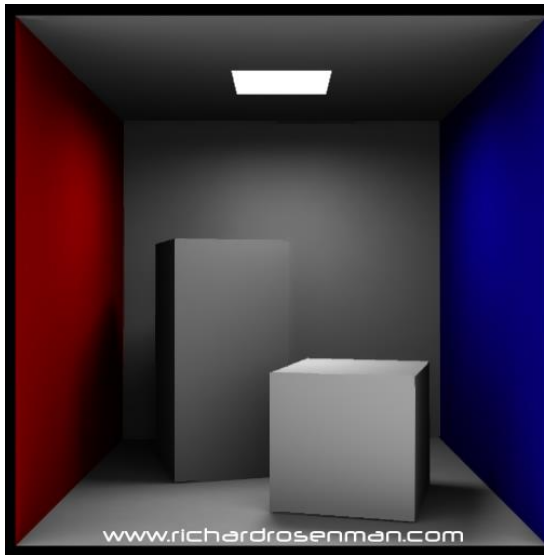
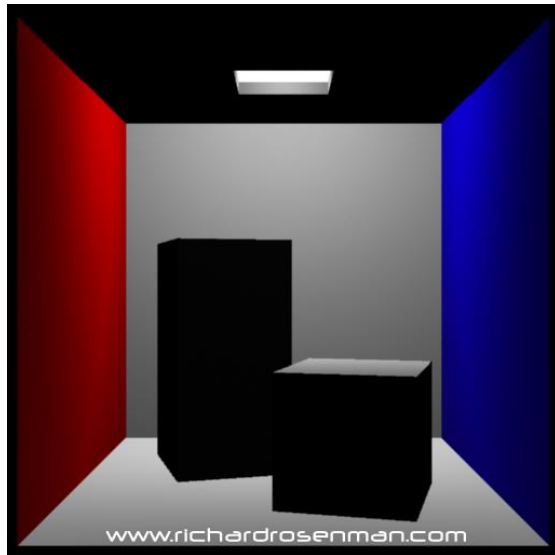
Ambient Occlusion



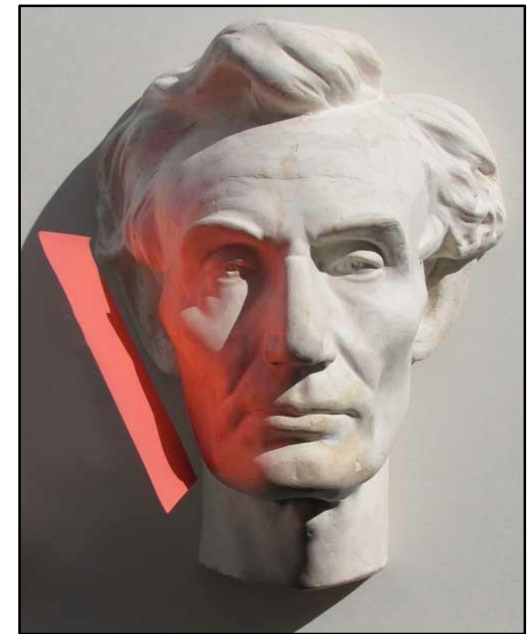
rendering a particle dataset without (top) and with (bottom) ambient occlusion
© Sebastian Grottel

Global Illumination

Multiple Scattering



increasing number of
diffuse surface scattering:
top left: *once*
top right: *twice*
bottom left: *infinite*



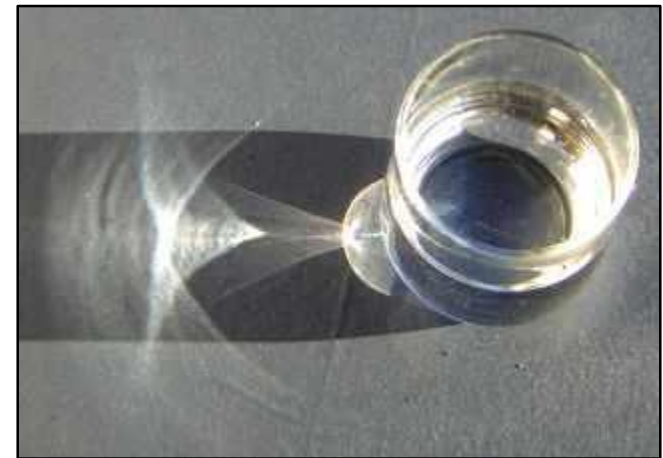
example of color bleeding
© James Gurney

Global Illumination

Mirror Reflection and Caustics



example of mirror reflections and soft shadows
© Prairie Light Images



example of caustic where light is bundled by concave curved refractive or mirror reflective surfaces, © Heiner Otterstedt



Crepuscular Rays are volumetric scattering effects for back light illumination
© Wikipedia



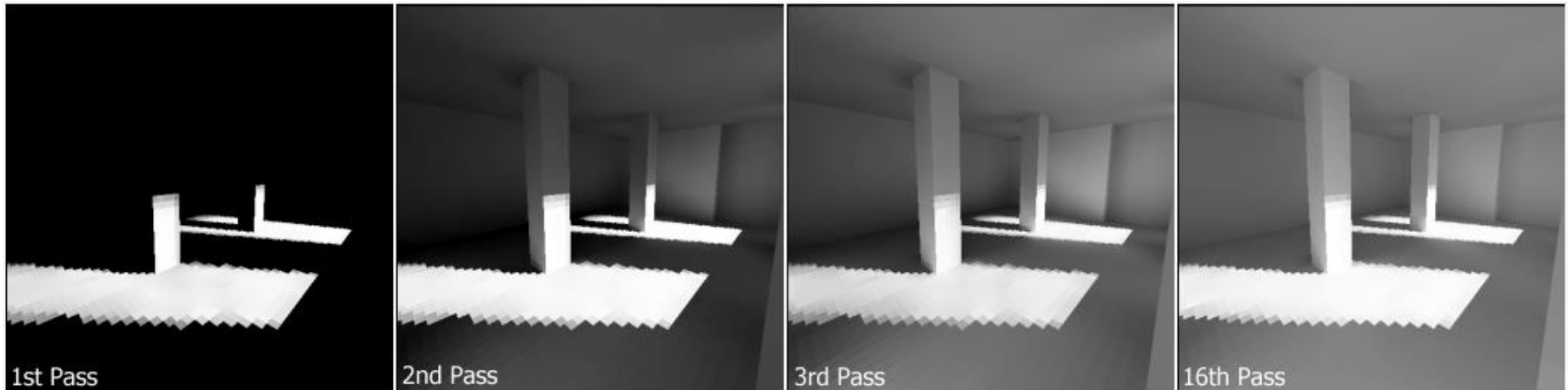
Realtime volumetric scattering in game engine,
© Markus Billeter et al 2012



Overview

The Rendering Equation

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- ◆ light transport and rendering equation
- ◆ basic radiometry and hemispherical reflectance
- ◆ numerical solution strategies



© Wikipedia – Radiosity, [link](#)

- ◆ light is reflected between mutually visible surfaces
- ◆ objects reflect and become emitters too
- ◆ **dynamic energy equilibrium**

emitted photons = absorbed photons + escaped photons

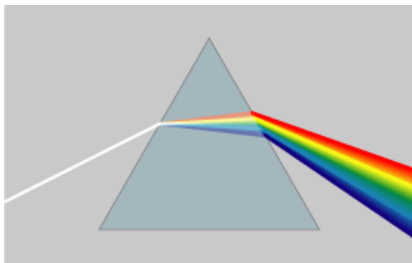
Light Transport

Setting for Opaque Scenes

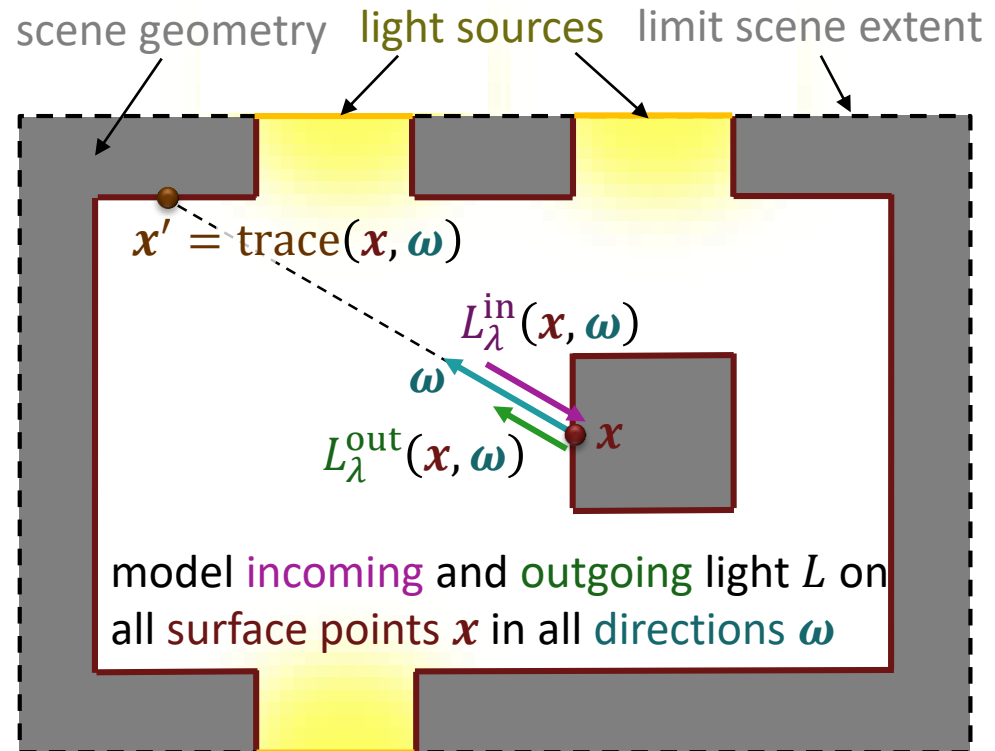


Typical Simplifications

- ◆ macroscopic
- ◆ geometric optics (linear propagation)
- ◆ no volumetric scattering
- ◆ no dispersion



- ◆ ignore polarization



model incoming and outgoing light L on all surface points $\mathbf{x}$ in all directions ω

linking of incoming and outgoing light with raytracing operator:

$$L_{\lambda}^{\text{in}}(\mathbf{x}, \omega) = L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \omega), -\omega)$$



- total outgoing light is emission plus reflection

$$L_{\lambda}^{\text{out}} = L_{\lambda}^{\text{emit}} + L_{\lambda}^{\text{reflect}}$$

$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) + \underline{L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}})}$$

- reflect incoming light from all directions with BRDF ρ

$$\underline{L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}})} = \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos\theta^{\text{in}} d\Omega^{\text{in}}$$



- link outgoing to incoming light with trace

$$L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) = L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}), -\boldsymbol{\omega}^{\text{in}})$$

$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) +$$

$$\iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}), -\boldsymbol{\omega}^{\text{in}}) \cos\theta^{\text{in}} d\Omega^{\text{in}}$$

directional formulation of *rendering equation*, which is also called *transport equation*

Rendering Equation

Directional Form [Immel et al 1986]



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) + \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}), -\boldsymbol{\omega}^{\text{in}}) \cos\theta^{\text{in}} d\Omega^{\text{in}}$$

directional formulation of *rendering equation*, which is also called *transport equation*

Parameters

- $\mathbf{x}$... surface point
- $\boldsymbol{\omega}^{\text{out}}$... outgoing direction
- $\boldsymbol{\omega}^{\text{in}}$... incoming direction
- θ^{in} ... angle between surface normal and incoming direction
- Ω^{in} ... outer hemisphere of surface is integration domain for incoming directions

Functions

- L_{λ}^{out} ... total outgoing radiance and to be solved for (*[Fredholm integral equation of 2nd kind](#)*)
- $L_{\lambda}^{\text{emit}}$... emitted radiance
- ρ ... bidirectional reflectance distribution function (BRDF)
- *trace*... raytracing operator

Rendering Equation

Important Topics related to RenEq



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \omega^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \omega^{\text{out}}) + \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \omega^{\text{in}}), -\omega^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$$

radiometry

- How to parametrize and integrate directions ω ?
- How to define $L_{\lambda}(\mathbf{x}, \omega)$ and $\rho(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}})$?

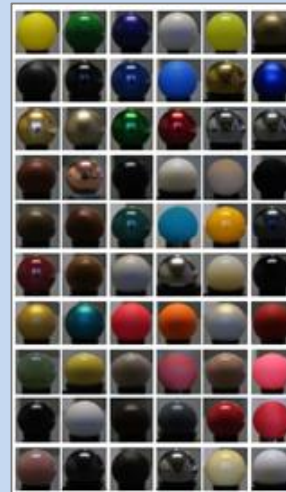
photometry

- How do we perceive light?

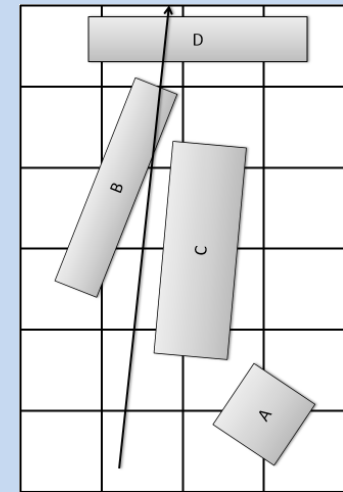
modeling light sources



modeling materials



ray-scene-intersection



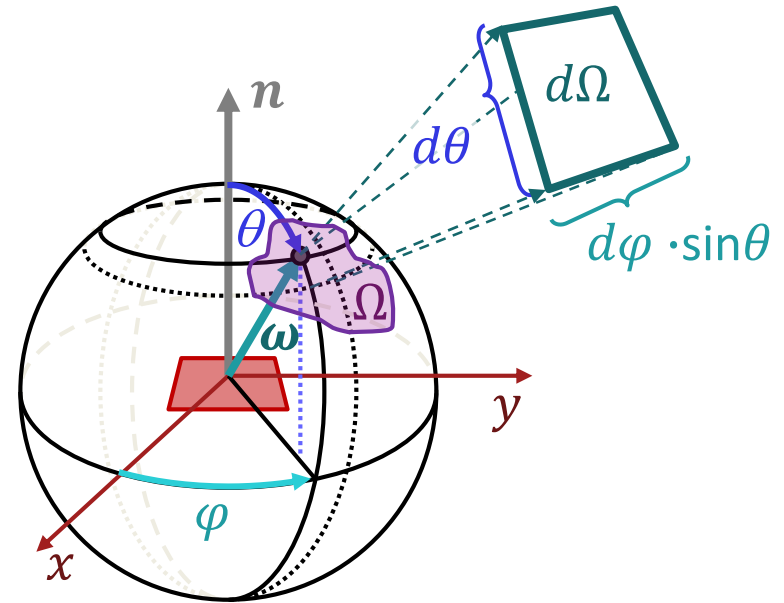
Directions and Solid Angle

Local Parametrization & Integration



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) + \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos\theta^{\text{in}} d\Omega^{\text{in}}$$

- direction $\boldsymbol{\omega}$ is parametrized over *unit sphere* in spherical coordinates φ, θ
- parametrization is relative to local surface normal $\mathbf{n}$
- solid angle Ω is measured in area covered on sphere
- SI unit: *sr* (steradian)
- integration of solid angle yields double integral over φ and θ



$$\boldsymbol{\omega} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}, \quad \begin{aligned} \varphi &\in [-\pi, \pi] \\ \theta &\in [0, \frac{\pi}{2}] \end{aligned}$$

$$d\Omega = d\varphi \cdot \sin\theta d\theta$$

Directions and Solid Angle

Example Integration



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) + \iint_{\Omega^{\text{in}}} \underbrace{\rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}})}_C \cos\theta^{\text{in}} d\Omega^{\text{in}}$$

$$\begin{aligned} \iint_{\Omega^{\text{in}}} C \cos\theta^{\text{in}} d\Omega^{\text{in}} &= C \int_0^{\frac{\pi}{2}} \left(\int_{-\pi}^{\pi} \cos\theta^{\text{in}} d\varphi \right) \cdot \sin\theta^{\text{in}} d\theta^{\text{in}} \\ &= C \int_0^{\frac{\pi}{2}} \left(\int_{-\pi}^{\pi} d\varphi \right) \cos\theta^{\text{in}} \cdot \sin\theta^{\text{in}} d\theta^{\text{in}} \\ &= 2\pi C \int_0^{\frac{\pi}{2}} \cos\theta^{\text{in}} \cdot \sin\theta^{\text{in}} d\theta^{\text{in}} \\ &= \pi C \left[-\cos^2\theta^{\text{in}} \right]_0^{\frac{\pi}{2}} = \pi C \end{aligned}$$

Directions and Solid Angle

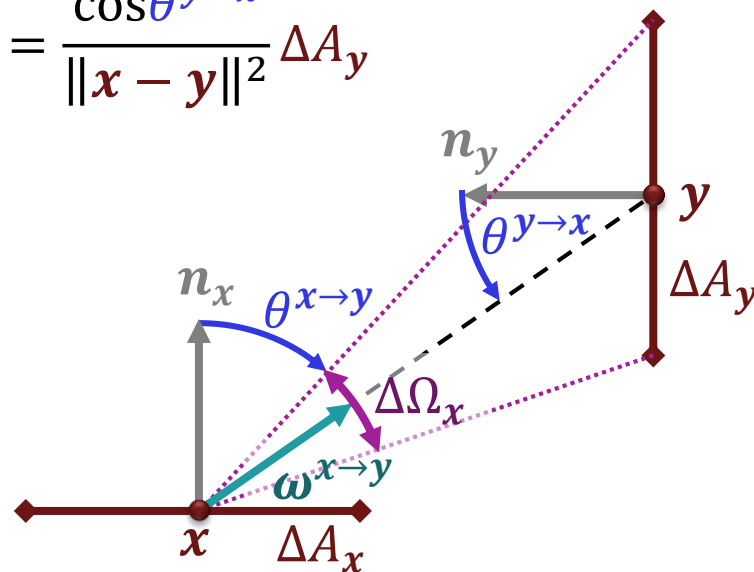
Connection to Surface Patches



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \omega^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \omega^{\text{out}}) + \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \omega^{\text{in}}) \underline{\cos\theta^{\text{in}} d\Omega^{\text{in}}}$$

- patch ΔA_y is seen from patch ΔA_x under the solid angle $\Delta\Omega_x$ that decreases with distance squared and cosine to normal of patch around $\mathbf{y}$
- plugging into rendering equation yields new form

$$\Delta\Omega_x = \frac{\cos\theta^{y \rightarrow x}}{\|\mathbf{x} - \mathbf{y}\|^2} \Delta A_y$$



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \omega^{\text{out}}) = L_{\lambda}^{\text{emit}} + \iint_{y \in A} \rho(\mathbf{x}, \omega^{x \rightarrow y}, \omega^{\text{out}}) L_{\lambda}^{\text{out}}(\mathbf{y}, \omega^{y \rightarrow x}) V(\mathbf{x}, \mathbf{y}) \frac{\cos\theta^{x \rightarrow y} \cos\theta^{y \rightarrow x}}{\|\mathbf{x} - \mathbf{y}\|^2} dA_y$$

rendering equation, with integration over scene area and visibility check

Rendering Equation II

Area Formulation [Kajiya et al 1986]



$$L_{\lambda}^{\text{out}}(\mathbf{x}, \omega^{\text{out}}) = L_{\lambda}^{\text{emit}}(\mathbf{x}, \omega^{\text{out}}) + \iint_{\mathbf{y} \in A} \rho(\mathbf{x}, \omega^{\mathbf{x} \rightarrow \mathbf{y}}, \omega^{\text{out}}) L_{\lambda}^{\text{out}}(\mathbf{y}, \omega^{\mathbf{y} \rightarrow \mathbf{x}}) G(\mathbf{x}, \mathbf{y}) dA_{\mathbf{y}}$$

area formulation of *rendering equation*

- integration of $\mathbf{y}$ over complete scene area A
- geometry factor $G(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) \frac{\cos \theta^{\mathbf{x} \rightarrow \mathbf{y}} \cos \theta^{\mathbf{y} \rightarrow \mathbf{x}}}{\|\mathbf{x} - \mathbf{y}\|^2}$
- when combining G with $dA_{\mathbf{y}}$ the units combine to sr:
$$[G \cdot dA_{\mathbf{y}}] = \frac{1}{m^2} \cdot m^2 = \frac{m^2}{m^2} = sr.$$
- with visibility test $V(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \dots \text{if visible} \\ 0 & \dots \text{if blocked} \end{cases}$

Rendering Equation III

Operator Formulation



- To gain insight on a higher level, one can define ray **tracing** and **reflection** as **functionals** or **operators** **T** and **R** that map radiance functions onto radiance functions:
- **$T : L_{\lambda}^{\text{out}} \rightarrow L_{\lambda}^{\text{in}}$** , **$L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}) = L_{\lambda}^{\text{out}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}), -\boldsymbol{\omega})$**
- **$R : L_{\lambda}^{\text{in}} \rightarrow L_{\lambda}^{\text{reflect}}$** , **$L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = \iint_{\Omega^{\text{in}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$**
- One can interpret the action of operators on functions like infinite dimensional matrices acting on infinite dimensional vectors, yielding the **operator form** of the rendering equation:

$$L_{\lambda}^{\text{out}} = L_{\lambda}^{\text{emit}} + \mathbf{R} \cdot \mathbf{T} \cdot L_{\lambda}^{\text{out}}$$

- solving for **L_{λ}^{out}** formally with Taylor expansion yields
$$L_{\lambda}^{\text{out}} = (\mathbf{I} - \mathbf{R} \cdot \mathbf{T})^{-1} L_{\lambda}^{\text{emit}} = L_{\lambda}^{\text{emit}} + \mathbf{RT} \cdot L_{\lambda}^{\text{emit}} + (\mathbf{RT})^2 \cdot L_{\lambda}^{\text{emit}} + \dots$$
- Thus the radiance function can be computed as the sum over emitted light, single reflection of emitted light, double reflection of emitted light and so on...
- This solution of 2nd kind Fredholm integral equations is called Liouville–Neumann series

Rendering Equation III

Operator Formulation Refined



- According to the splitting of the BRDF into diffuse ($\mathbf{D}$), specular ($\mathbf{S}$) and mirror reflecting ($\mathbf{M}$) part, one can refine the combined trace-reflection operator:

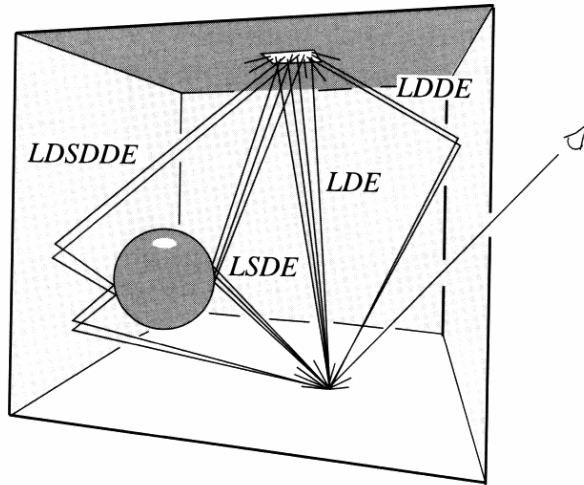
$$\mathbf{RT} = \mathbf{D} + \mathbf{S} + \mathbf{M}$$

- Refinement of formal solution yields

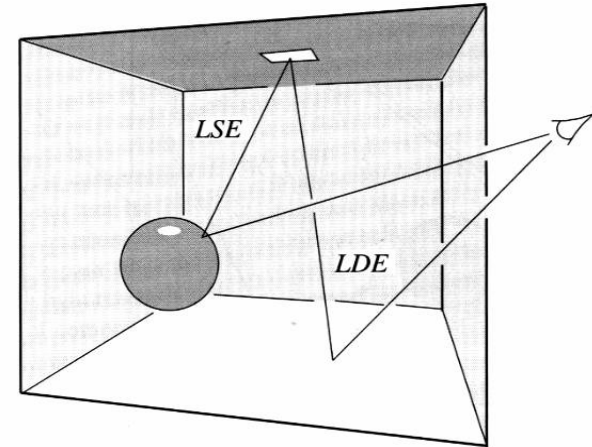
$$L_{\lambda}^{\text{out}} = L_{\lambda}^{\text{emit}} + (\mathbf{D} + \mathbf{S} + \mathbf{M}) \cdot L_{\lambda}^{\text{emit}} + (\mathbf{D} + \mathbf{S} + \mathbf{M})^2 \cdot L_{\lambda}^{\text{emit}} + \dots$$

- If L_{λ}^{out} is received in eye, this means that the rendering equation is solved by adding up contributions by all different kinds of ray paths.
- The paths are typically denoted in **order of light transport** from light source (L) to eye (E): $LDDSM E$
- One uses **regular expression** syntax to capture sets of rays: $L(D|S)^*E$

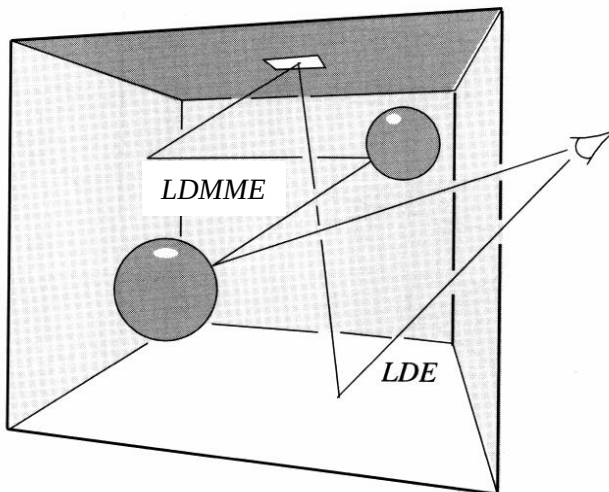
Operator Formulation Examples



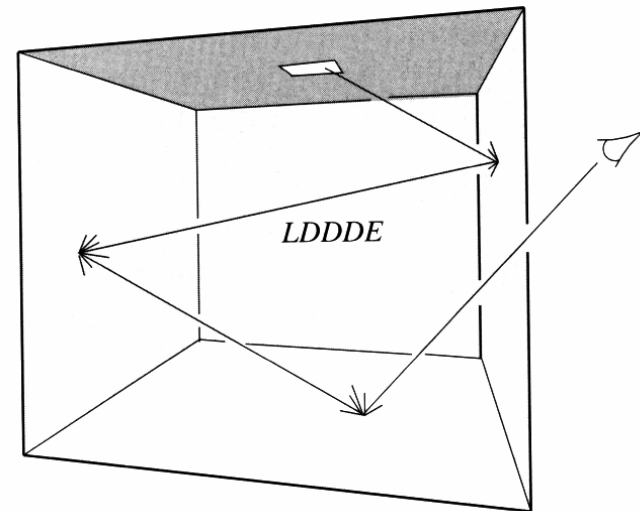
all paths: $L(D|S|M)*E$



local illumination: $L(D|S)E$



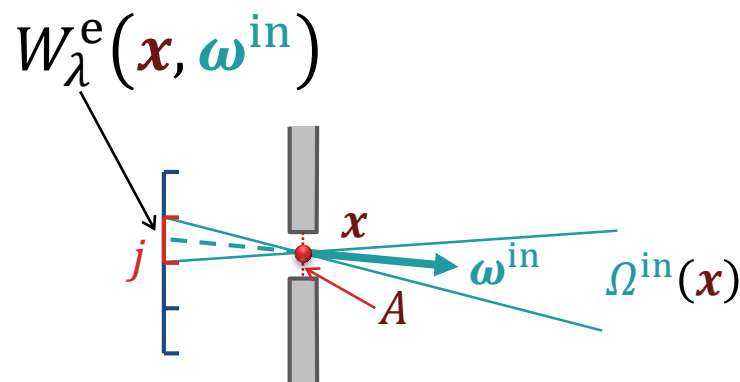
raytracing: $L(S|D)M*E$



radiosity: $LD*E$

Sensor Efficiency

- dual to light sources the human eye or a camera acts as light sensor
- The function $W_\lambda^e(\mathbf{x}, \omega^{\text{in}})$ defines the **spectral efficiency** of the sensor for an incoming ray.
- Efficiency is the percentage of radiance, which was detected
- In the simplest case efficiency is one for rays that hit the sensor and zero otherwise.
- The brightness of the pixel is spectral power and computed through the **measurement equation** as integral over all incoming rays hitting the sensor.



simplified camera model with sample point $\mathbf{x}$ on aperture of area A and solid angle of incoming directions that hit sample pixel j

$$\Phi_{\lambda,j} = \iint_A \iint_{\Omega_j^{\text{in}}(\mathbf{x})} W_\lambda^e(\mathbf{x}, \omega^{\text{in}}) L_\lambda^{\text{in}}(\mathbf{x}, \omega^{\text{in}}) \cos\theta^{\text{in}} d\Omega_j^{\text{in}} dA$$

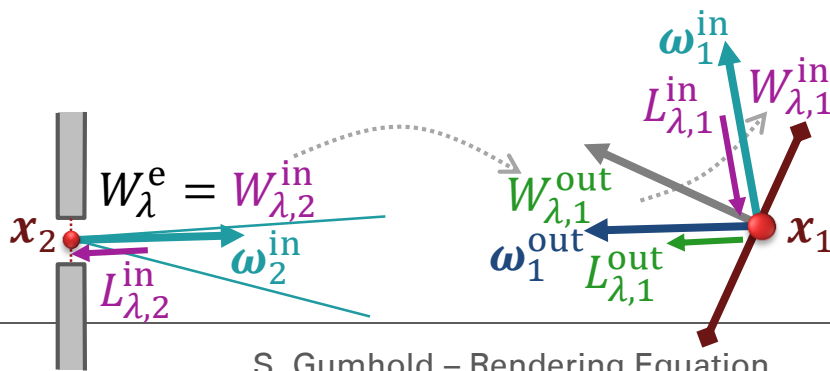
measurement equation

Importance Function

- To steer the integration of the rendering equation one defines the spectral **incoming** and **outgoing importance function** (also called **potential** or **response function**) $W_\lambda^{\text{in}} / W_\lambda^{\text{out}}$.
- $W_\lambda^{\text{in}} / W_\lambda^{\text{out}}$ measures the influence of the radiance $L_\lambda^{\text{in}} / L_\lambda^{\text{out}}$ onto the sensor (often restricted to one pixel)
- based on the sensor efficiency function W^e importance is computed as follows and flows from the sensor to the light sources

$$W_\lambda^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) = W_\lambda^e(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) + \iint_{\Omega^{\text{out}}} \rho(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}, \boldsymbol{\omega}^{\text{in}}) W_\lambda^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}}$$

spectral importance function $W_\lambda^{\text{out}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = W_\lambda^{\text{in}}(\text{trace}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}), -\boldsymbol{\omega}^{\text{out}})$





- ◆ The rendering equation combines light transport and equilibrium of light emission and absorption processes
- ◆ There are two parameterizations of the rendering equation: over directions and over surface area
- ◆ The operator notation allows to formally solve the rendering equation by adding contributions of all types of light paths
- ◆ Sensor efficiency is defined per ray that hits sensors
- ◆ Measurements are integrated over ray space and measured in units of power
- ◆ Importance function is used to measure influence of radiance along an arbitrary ray in the scene on the measurement with a sensor



Overview

The Rendering Equation

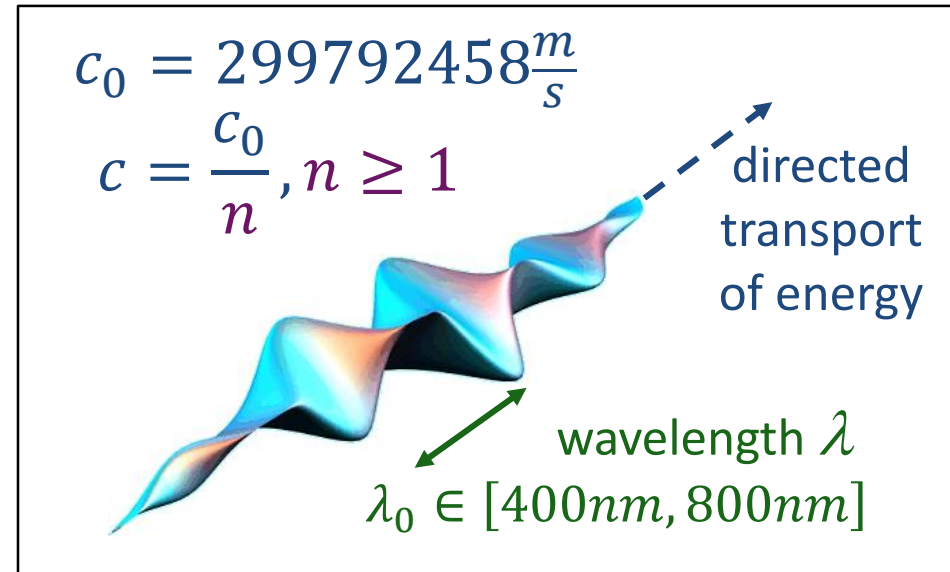
- ◆ motivation and simplifications
- ◆ light transport and rendering equation
- ◆ basic radiometry and hemispherical reflectance
- ◆ numerical solution strategies

Radiometry

Basic Properties of Light



- light is composed of **photons**
- photons travel with the speed of light c
- in medium slower by **index n of refraction**
- each photon has characteristic **frequency ν** from which the **wavelength λ** can be computed based on speed c in medium.
- frequency also defines **energy q** carried by a photon



visualization of a light photon, © 1988 by the
TAT Foundation, [link](#)

$$c = \nu \cdot \lambda$$

$$\nu \in [3.75 \cdot 10^{14} \text{ Hz}, 7.5 \cdot 10^{14} \text{ Hz}]$$

$$q = h \cdot \nu, h = 6.626 \cdot 10^{-34} \text{ Js}$$

Planck constant

$$q \in [2.5 \cdot 10^{-19} \text{ J}, 5.0 \cdot 10^{-19} \text{ J}]$$

Radiometry

Spectral Energy Density



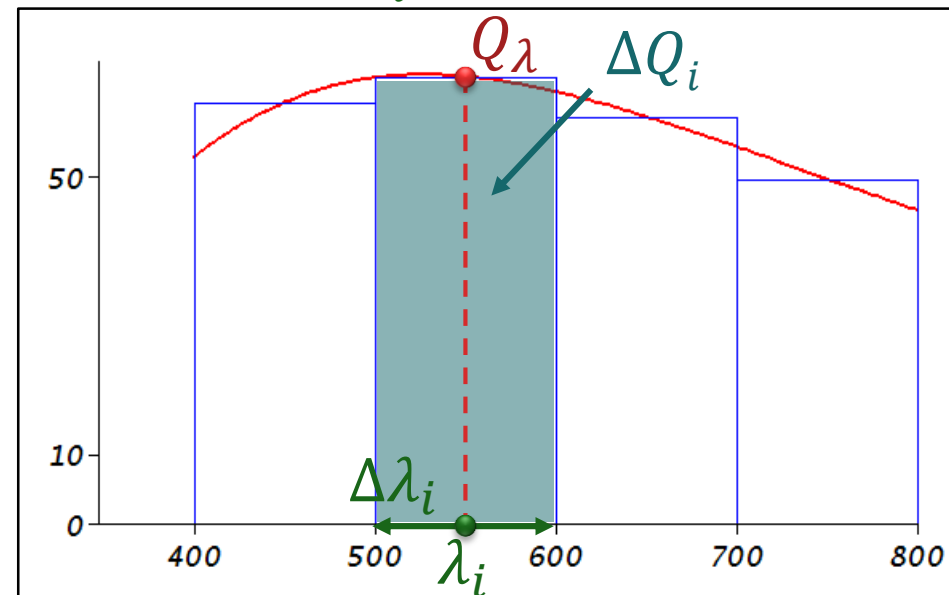
- spectral filter allows to measure photon energy ΔQ_i within wavelength interval $\Delta\lambda_i$ around λ_i
- relating ΔQ_i to the interval width yields the spectral energy density $Q_\lambda(\lambda_i)$
- with narrower filters spectral energy density converges to derivative

$$Q_\lambda(\lambda) = \frac{dQ}{d\lambda}$$

$$Q = \int_{400nm}^{800nm} Q_\lambda(\lambda) d\lambda \approx \sum_i \Delta Q_i \cdot \Delta\lambda_i$$

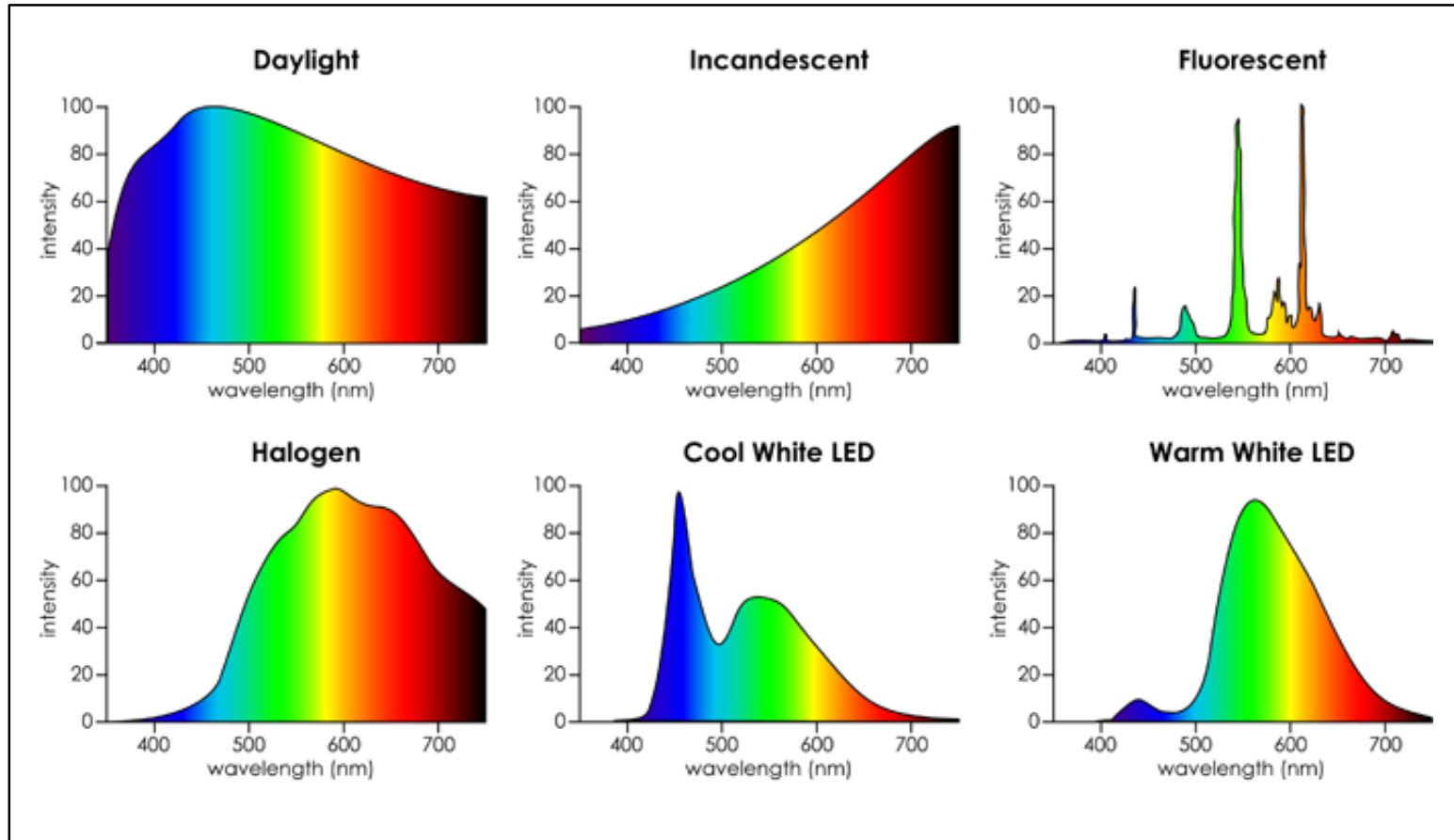
$$\Delta Q_i = \sum_{j|\lambda_j \in \Delta\lambda_i} q_j$$

$$Q_\lambda(\lambda_i) \approx \frac{\Delta Q_i}{\Delta\lambda_i}$$



Radiometry

Spectral Energy Density



emission spectrum of different light sources, © [housecraft](#)

Radiometry

Power and Irradiance

- spectral power Φ_λ is density of spectral energy Q_λ in time t

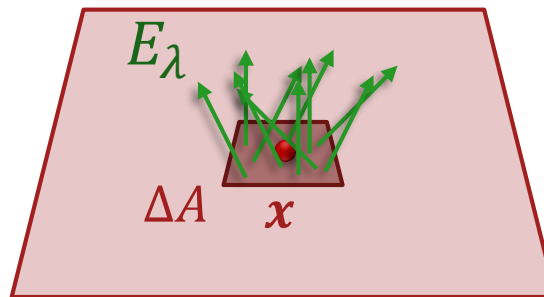
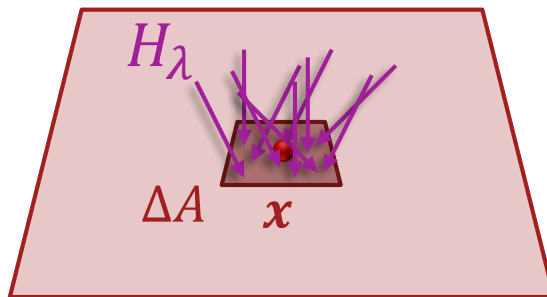
$$\Phi_\lambda(\lambda, t) = \frac{dQ_\lambda}{dt}$$

$$Q_\lambda(\lambda) = \int \Phi_\lambda(\lambda, t) dt \approx \sum \Phi_{\lambda,i} \Delta t_i$$

- irradiance H_λ / radiant exitance or radiosity E_λ are densities of incoming / emitted spectral power in surface area A

$$H_\lambda | E_\lambda(\mathbf{x}) = \frac{d\Phi_\lambda^{\text{in|out}}}{dA}$$

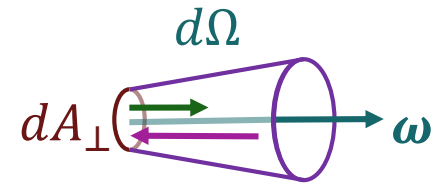
$$\Phi_\lambda^{\text{in}} = \iint H_\lambda(\mathbf{x}) dA \approx \sum H_{\lambda,i} \Delta A_i$$



$$dA = dx \cdot dy$$

$$\Delta A = \Delta x \cdot \Delta y$$

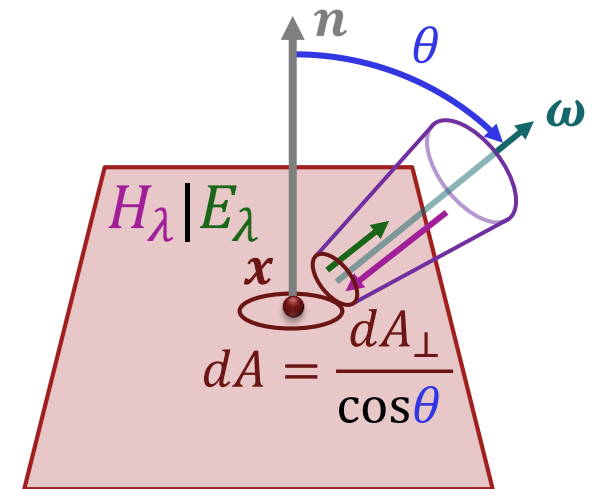
- incoming and outgoing radiance $L_\lambda^{\text{in|out}}$ are densities of $\Phi_\lambda^{\text{in|out}}$ over the direction ω of transport and the area $A_\perp$ **orthogonal** to ω .



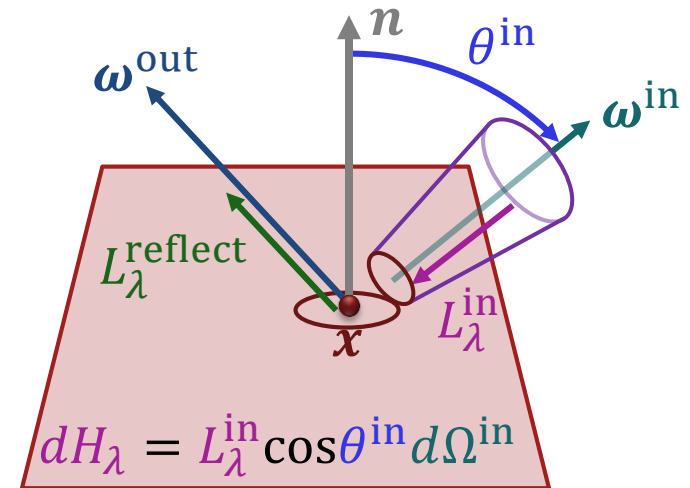
$$L_\lambda^{\text{in|out}}(\mathbf{x}, \omega) = \frac{d^2 \Phi_\lambda^{\text{in|out}}}{dA_\perp d\Omega^{\text{in|out}}}$$

- in vacuum *radiance is constant along ray*
- carfull:** relating to surface area needs additional cosine term:

$$H_\lambda | E_\lambda(\mathbf{x}) = \frac{d\Phi_\lambda}{dA} = \frac{\cos\theta \cdot d\Phi_\lambda}{dA_\perp} = \iint_{\Omega} L_\lambda(\mathbf{x}, \omega) \cos\theta d\Omega$$



- surface reflection distributes **incoming light** over all **outgoing** directions
- the bidirectional reflectance distribution function ρ (BRDF) is density of **reflected radiance** $L_{\lambda}^{\text{reflect}}$ with respect to **irradiance** H_{λ}
- integration over hemisphere yields total reflected radiance



$$\rho_{\lambda}(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) = \frac{dL_{\lambda}^{\text{reflect}}(\mathbf{x}, \omega^{\text{out}})}{dH_{\lambda}(\mathbf{x}, \omega^{\text{in}})}$$

$$L_{\lambda}^{\text{reflect}}(\mathbf{x}, \omega^{\text{out}}) = \iint_{\Omega^{\text{in}}} \rho_{\lambda}(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \omega^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$$

Radiometry

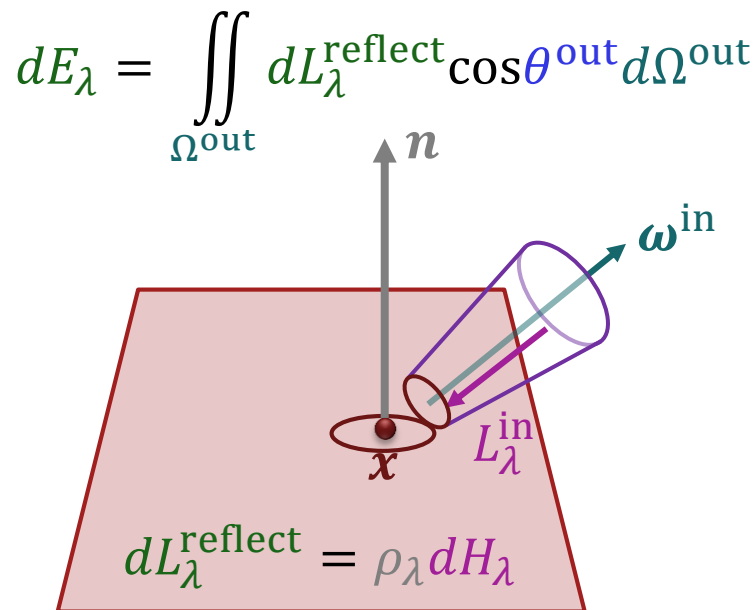
Diffuse BDRF & Reflectance



- directional hemispherical reflectance R_λ or albedo computes fraction of outgoing light over incoming light focused around ω^{in}

$$R_\lambda(\omega^{\text{in}}) = \frac{dE_\lambda(\omega^{\text{in}})}{dH_\lambda(\omega^{\text{in}})} \in [0,1]$$

$$= \iint_{\Omega^{\text{out}}} \rho_\lambda(\mathbf{x}, \omega^{\text{in}}, \omega^{\text{out}}) \cos\theta^{\text{out}} d\Omega^{\text{out}}$$



- for a constant (diffuse) BRDF, also R_λ is constant

$$R_\lambda(\omega^{\text{in}}) = \rho_\lambda^{\text{diff}} \iint_{\Omega^{\text{out}}} \cos\theta^{\text{out}} d\Omega^{\text{out}} = \pi \rho_\lambda^{\text{diff}}$$

→ R_λ defines $\rho_\lambda^{\text{diff}}$ fully:

$$\rho_\lambda^{\text{diff}} = R_\lambda / \pi$$

Radiometry

Diffuse BDRF & Reflectance



- Full derivation of formula for directional hemispherical reflectance

$$E_{\lambda}(\mathbf{x}) = \iint_{\Omega^{\text{out}}} L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}}$$

$$L_{\lambda}^{\text{reflect}}(\mathbf{x}, \boldsymbol{\omega}^{\text{out}}) = \iint_{\Omega^{\text{in}}} \rho_{\lambda}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}$$

$$E_{\lambda}(\mathbf{x}) = \iint_{\Omega^{\text{in}}} \left[\iint_{\Omega^{\text{out}}} \rho_{\lambda}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}} \right] \underbrace{L_{\lambda}^{\text{in}}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}) \cos \theta^{\text{in}} d\Omega^{\text{in}}}_{dH_{\lambda}}$$

$$E_{\lambda}(\mathbf{x}) = \iint_{\Omega^{\text{in}}} \left[\iint_{\Omega^{\text{out}}} \rho_{\lambda}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}} \right] dH_{\lambda}$$

$$R_{\lambda}(\boldsymbol{\omega}^{\text{in}}) = \frac{dE_{\lambda}(\boldsymbol{\omega}^{\text{in}})}{dH_{\lambda}(\boldsymbol{\omega}^{\text{in}})} = \iint_{\Omega^{\text{out}}} \rho_{\lambda}(\mathbf{x}, \boldsymbol{\omega}^{\text{in}}, \boldsymbol{\omega}^{\text{out}}) \cos \theta^{\text{out}} d\Omega^{\text{out}}$$



- ◆ radiometric quantities are derived from photon energy per time (power) and the power densities with respect to area and or direction
- ◆ most important is radiance – the light power travelling along a ray. Without volumetric scattering, the radiance is constant along rays.
- ◆ The BRDF relates incoming irradiance with outgoing radiance and is integrated over hemisphere relative to surface normal.
- ◆ BRDFs can be analyzed by integrating over one directional argument. Integration over outgoing directions yields directional hemispherical reflections that must be ≤ 1 .



Overview

The Rendering Equation

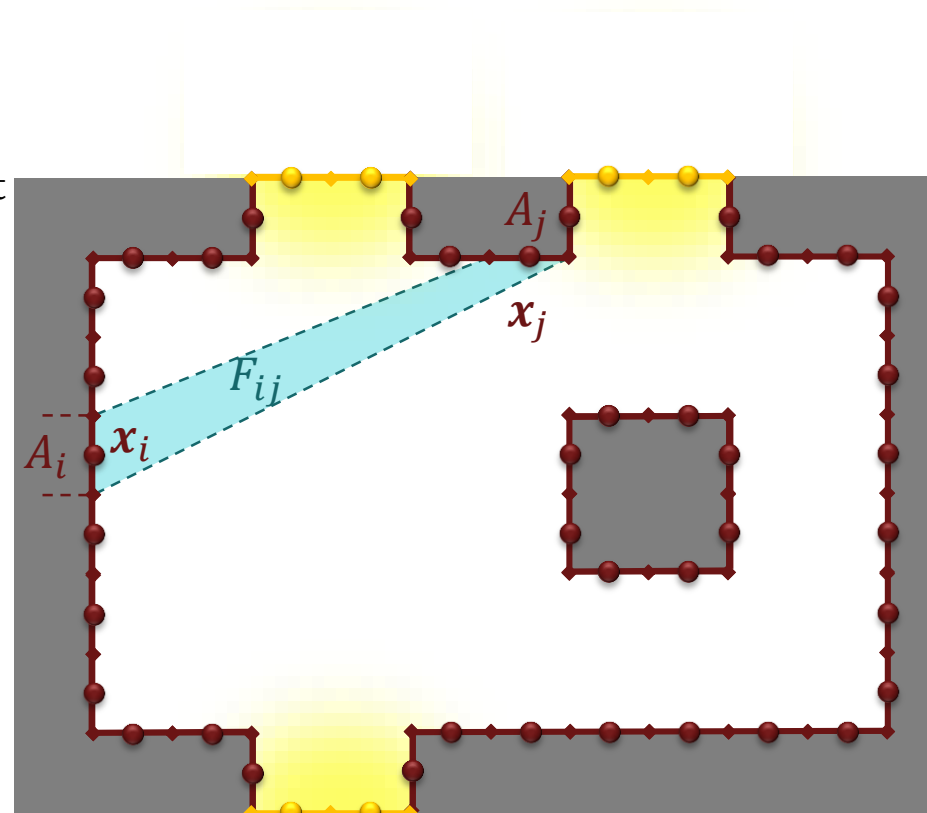
- ◆ motivation and simplifications
- ◆ light transport and rendering equation
- ◆ basic radiometry and hemispherical reflectance
- ◆ numerical solution strategies

Solving the Rendering Equation

Radiosity



- ◆ **Assumption:** only **diffuse** emission and reflection
- ◆ dependency on direction can be dropped on outgoing radiance L_{out} and restriction to radiosity E possible
- ◆ discretize surface into patches A_i around $\mathbf{x}_i$
- ◆ integrate pairwise geometry terms F_{ij} over patches A_i and A_j
- ◆ derive system of linear equations:
$$E_{\lambda,i}^{\text{out}} = E_{\lambda,i}^{\text{emit}} + \sum_j \rho_i F_{ij} E_{\lambda,j}^{\text{out}}$$
- ◆ and solve for vector of radiosities $E_{\lambda,i}^{\text{out}}$
- ◆ solution is independent of view point and can be baked into scene



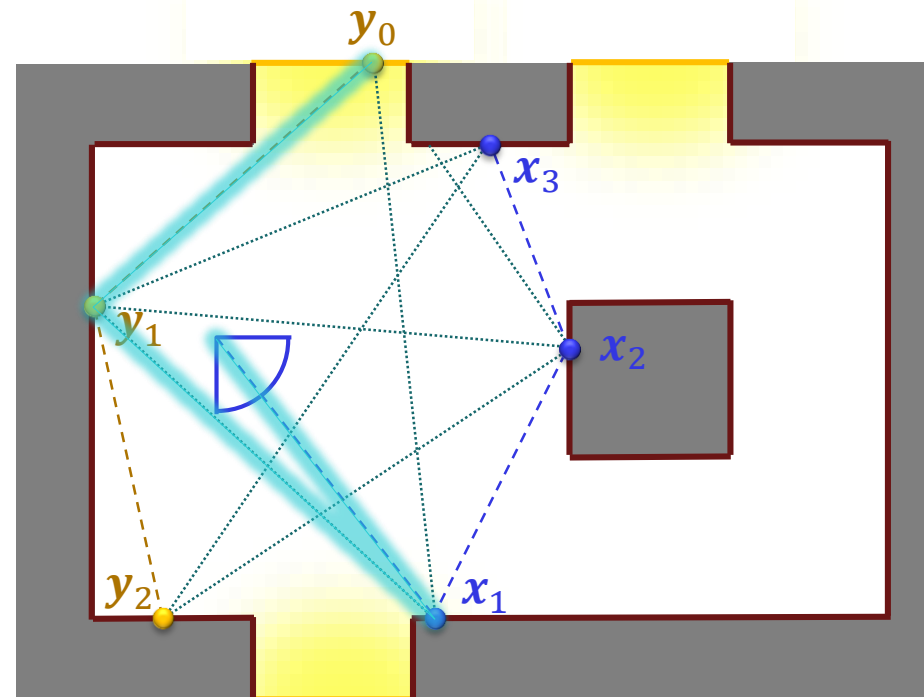
Solving the Rendering Equation

Bidirectional Path-Tracing



- define camera and repeat multiple times per pixel
- sample camera path
 - sample primary ray and find intersection x_1
 - while Russian roulette
 - sample secondary ray and find intersection x_i
- sample light path
 - sample light position y_0
 - while Russian roulette
 - sample secondary ray and find intersection y_i
- interconnect paths

Monte-Carlo Integration



compare java-script Path-Tracing under
<http://madebyevan.com/webgl-path-tracing>
and try to find a caustic



Summary

- ◆ light is modeled as continuous radiance fields on scene surface
- ◆ light bounces between visible surfaces until equilibrium
- ◆ bounces are described by BRDF (material property)
- ◆ rendering equation is integral equation in outgoing radiance (or optionally in incoming radiance)
- ◆ pixel values are computed by integration of radiance over the rays hitting the sensor
- ◆ solution strategies build on discretizations of the scene or monte-carlo integration techniques
- ◆ assumption of pure diffuse emission and reflection reduces problem to a system of linear equations



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