

# Novel routes beyond the Landau-Ginzburg-Wilson paradigm

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# Novel routes beyond the Landau-Ginzburg-Wilson paradigm

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# Abstract

Even a century after Heisenberg, Born, Jordan, and others laid the foundations of modern quantum mechanics, quantum phenomena continue to fascinate and puzzle us, with no end to their mysteries in sight. A particularly rich and active field, grounded in a variety of deeply rooted quantum-mechanical concepts, is that of quantum phase transitions, i.e., phase transitions at zero temperature controlled by nonthermal parameters. While a remarkable number of quantum phase transitions are successfully described within the Landau-Ginzburg-Wilson paradigm, particular attention is drawn to cases where a description within the paradigm fails. Prominent examples that evade the Landau-Ginzburg-Wilson paradigm are phase transitions beyond conventional symmetry breaking and phase transitions enriched by additional gapless modes. This dissertation seeks to further advance the theory of phase transitions beyond the Landau-Ginzburg-Wilson paradigm by proposing and analyzing several novel routes beyond the framework and relating them to established concepts in the field.

First, we identify an exotic quasiuniversal behavior near the all-in-all-out Weyl quantum critical point in three-dimensional Luttinger semimetals, such as the pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ , with  $R$  a rare-earth element. The quasiuniversal behavior is characterized by power laws with exponents that vary slowly over several orders of magnitude in energy or length. However, in contrast to the quasiuniversality discussed in the context of deconfined criticality, the present case is characterized by a genuinely universal ultra-low-temperature behavior. In this limit, the pertinent critical exponents can be computed exactly within a renormalization group analysis. This exotic behavior will manifest generically at any quantum critical fixed point possessing a marginally irrelevant operator. Experimental implications for the pyrochlore iridates are outlined and compared with recent experimental observations in  $\text{Y}_2\text{Ir}_2\text{O}_7$  thin films.

Second, we propose a novel mechanism for continuous order-to-order quantum phase transitions that, unlike deconfined criticality, operates independently of fractionalization. The proposed mechanism is based on the collision and annihilation of two renormalization group fixed points: a quantum critical fixed point and an infrared stable fixed point. The annihilation of these fixed points rearranges the flow topology, eliminating the disordered phase associated with the infrared stable fixed point and promoting a second critical fixed point, unaffected by the collision, to a quantum critical point between distinct orders. We argue that this mechanism is relevant to a broad spectrum of physical systems. In particular, it can manifest in Luttinger fermion systems in three spatial dimensions, leading to a continuous quantum phase transition between an antiferromagnetic Weyl semimetal

state, which breaks time-reversal symmetry, and a nematic topological insulator, characterized by broken lattice rotational symmetry. This continuous antiferromagnetic-Weyl-to-nematic-insulator transition might be observed in rare-earth pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ . Other possible realizations include kagome quantum magnets, quantum impurity models, and quantum chromodynamics with supplemental four-fermion interactions. This broad spectrum of potential realizations provides an ideal setting to explore the proposed fixed-point-annihilation mechanism and its interrelations with other concepts, such as fractionalization.

Third, we present multiple strands of ongoing research, all ultimately working towards a putative Sachdev-Ye-Kitaev fixed point in three spatial dimensions. As a starting point, we investigate the large- $N$  behavior of a partially bosonized Luttinger fermion theory at a large but finite number of quadratic band touching points  $N$ . Building on this, we analyze a random variant of the deterministic Luttinger fermion model, now in the limit of infinitely many fermionic and bosonic degrees of freedom, motivated by low-dimensional Yukawa-Sachdev-Ye-Kitaev theories. In both cases, complications, ultimately resulting from the absence of Lorentz symmetry, arise and motivate the study of a slightly simpler analogue, a relativistic Luttinger fermion theory. Not only does this third study improve our understanding of Luttinger fermion systems at large  $N$  in general, but it also reveals a novel fermionic conformal fixed point, which corresponds to an infrared stable fixed point in  $\frac{8}{3} < D < 4$  space-time dimensions and to an ultraviolet stable fixed point in  $4 < D < 6$  space-time dimensions. While the former gives rise to a previously unknown conformal phase, the latter unveils a new asymptotically safe quantum field theory.

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# 1. Introduction

*“Most people who haven’t been trained in physics probably think of what physicists do as a question of incredibly complicated calculations, but that’s not really the essence of it. The essence of it is that physics is about concepts, wanting to understand the concepts, the principles by which the world works.” [1]*

— Edward Witten

Physics may be viewed as a closely intertwined web of principles, established by the combination of experimental evidence on one side and mathematical reasoning, or as Witten put it, “incredibly complicated calculations” [1], on the other side. As physicists, we strive to discover new, fundamentally different concepts that extend our web of principles and give us a more complete or a more precise picture of the world we live in. This is also the ultimate goal of this dissertation, to introduce and exemplify several novel principles in the context of phase transitions and relate them to existing concepts in the field. However, before exploring new principles, it is, of course, advisable to get familiar with the well-established concepts in the field of phase transitions first, which is the main purpose of this first chapter. In Sec. 1.1 we begin by introducing the Landau-Ginzburg-Wilson paradigm, a framework which has been successfully applied to a vast number and wide variety of thermal and quantum phase transitions. It is precisely this remarkable success that draws particular attention to phase transitions that defy a description within the paradigm. We provide a broad overview of several such classes, including phase transitions beyond conventional symmetry breaking in Sec. 1.2, phase transitions enriched by additional gapless modes in Sec. 1.3, and deconfined phase transitions in Sec. 1.4. Such phase transitions are of great interest, as they go beyond the established principles and ask, at least, for an extension of these principles or even for entirely new concepts. In Sec. 1.5, towards the end of this chapter, we provide an overview of the three research chapters of this dissertation, Chaps. 2–4, which present novel routes beyond the Landau-Ginzburg-Wilson paradigm. We take Witten’s quote above as a guiding principle for all chapters of this dissertation and aim to emphasize concepts rather than mathematical details, without sacrificing rigor.

## 1.1. Landau-Ginzburg-Wilson paradigm

In this section, we begin by reviewing the various areas of science and everyday life in which phase transitions can be encountered. Motivated by this overview, we present Landau-

Ginzburg theory, a mean-field approach to phase transitions, and illustrate it using the  $O(N)$  transition. Lastly, we incorporate fluctuation effects via a functional description of the problem and analyze the Wilson-Fisher fixed point in the Landau-Ginzburg-Wilson paradigm, which captures a remarkably large number of both thermal and quantum phase transitions. General concepts are again illustrated using the  $O(N)$  transition.

### 1.1.1. Motivation

We encounter phase transitions on a daily basis, for instance, the liquid-gas transition of water when cooking, the solid-liquid transition of water during snowmelt in spring, or the phase transitions of paraffin wax as candles burn. Phase transitions also play a crucial role across industry, technology, and healthcare. Notable examples include various solid-liquid and solid-solid transitions in the metal industry [2, 3], phase transitions in liquid crystal displays [4, 5], and the metal-to-superconductor transition of NbTi, Nb<sub>3</sub>Sn, or MgB<sub>2</sub> relevant for magnetic resonance imaging [6]. While superconducting transitions [7–12] are, of course, also relevant to fundamental physics, there are many other prominent examples, for instance, magnetic phase transitions [13–15], the Mott transition [16–19], and Bose-Einstein condensation [20–24] in statistical and condensed matter physics or the chiral phase transition [25–27] and electroweak symmetry breaking [28–30] in particle physics and cosmology. Moreover, phase transitions are also encountered beyond the realm of physics, for instance in biology, chemistry, game theory, and nonlinear dynamics. Examples are numerous, ranging from flocks of birds near criticality [31, 32] and instabilities in reaction-diffusion systems [33, 34] to topological phase transitions in predator-prey models [35] and universal behavior associated with the Kardar-Parisi-Zhang equation [36, 37]. Phase transitions can also arise from various forms of human interaction. Examples are, again, manifold and include phase transitions in traffic flows [38, 39], opinion dynamics [40], and stock markets [41]. Recently, it has been demonstrated that even deep neural networks are not immune to phase transitions [42]. In short, phase transitions are encountered in numerous research areas in the quantitative sciences.

### 1.1.2. Landau-Ginzburg theory

#### General concepts and thermal phase transitions

Having reminded ourselves of the remarkably broad scope of phase transitions, we now seek a general phenomenological framework that describes as many of them as possible. In the 1930s, Landau laid the foundation for such a framework by introducing the concept of a local order parameter  $\phi_a$ , which signals the disordered symmetric (ordered symmetry-broken) phase by a trivial (nontrivial) expectation value  $\langle\phi_a\rangle = 0$  for all components  $a$  ( $\langle\phi_a\rangle \neq 0$  for at least one component  $a$ ) [43]. The expectation value is determined by minimizing the Landau potential, assumed to be an analytic function of the order parameter  $\phi_a$  and to respect the symmetries of the system. Landau and Ginzburg later

generalized this framework by allowing for a space-dependent local order-parameter field  $\phi_a(\vec{x})$  [44]. To illustrate this framework further, let us take a look at the  $O(N) \rightarrow O(N-1)$  transition [45], for simplicity in  $d > 2$  ( $d > 1$ ) spatial dimensions for  $N > 1$  ( $N = 1$ ), cf. Sec. 1.2.1. The corresponding Landau-Ginzburg free energy, which generalizes the Landau potential, is given by [46, 47]

$$F_{\text{LG}} = \int d^d x \left[ \frac{1}{2} \phi_a (r - \nabla^2) \phi_a + \frac{\lambda}{4!} (\phi_a^2)^2 + J_a \phi_a \right], \quad (1.1)$$

where  $\phi_a \equiv \phi_a(\vec{x})$  is an unconstrained<sup>1</sup> real order-parameter field,  $J_a \equiv J_a(\vec{x})$  denotes a source field, and summation over repeated indices  $a = 1, \dots, N$  is assumed. Further,  $r$  denotes the tuning parameter and  $\lambda$  the quartic self-interaction. The above Landau-Ginzburg free energy, asymptotically exact in the vicinity of a critical point of a system in sufficiently high spatial dimensions, constitutes a truncated expansion in terms of powers and gradients of the bosonic order-parameter field  $\phi_a$ . By construction, the Landau-Ginzburg free energy density is not only a *local* quantity but also *analytic* in  $\phi_a$ . We retain only those terms in the expansion that are *invariant under all symmetry transformations*, specifically, global  $O(N)$  rotations in the present case, and add the symmetry-breaking source term by hand.

Above, we implicitly assumed that phase transitions can be described by specifying only a few key characteristics of a system, such as its symmetries or its dimension, while being insensitive to microscopic details. This assumption will be justified in the following subsection and implies that different microscopic models sharing the same key characteristics exhibit identical critical behavior, the notion of *universality*. As this introductory chapter is devoted to conceptual foundations, we would also like to add a few remarks here on the concept of spontaneous symmetry breaking and the related Goldstone theorem. The former refers to a phenomenon in many-body systems where the ground state acquires lower symmetry than the Landau-Ginzburg free energy [43, 48]. In case we spontaneously break a continuous symmetry  $G \rightarrow H$  to a subgroup  $H \subset G$ , the Goldstone theorem ensures the existence of  $\dim G - \dim H$  massless linearly dispersing bosonic modes, the (Nambu-)Goldstone bosons, universal features of symmetry-broken phases [49–51]. There exist various generalizations of the Goldstone theorem, for instance to nonrelativistic systems allowing for nonlinearly dispersing Goldstone modes [52], to systems with approximate symmetries resulting in pseudo-Goldstone modes with small but finite masses [53], to supersymmetric settings realizing fermionic Goldstone modes [54], or to higher-form symmetries, providing, for example, a modern explanation for the masslessness of the photon [55].

Let us return to the above theory, Eq. (1.1), which can be viewed as the saddle-point approximation of the functional integral representation of the partition function considered in the subsequent subsection. However, already at the mean-field level we encounter

<sup>1</sup>Rather than enforcing the hard constraint  $\phi_a^2 = 1$ , we choose to work with an unconstrained field subject to a potential in this section [46]. For a related problem where the hard constraint is enforced within a nonlinear sigma model description, see Sec. 1.4.3.

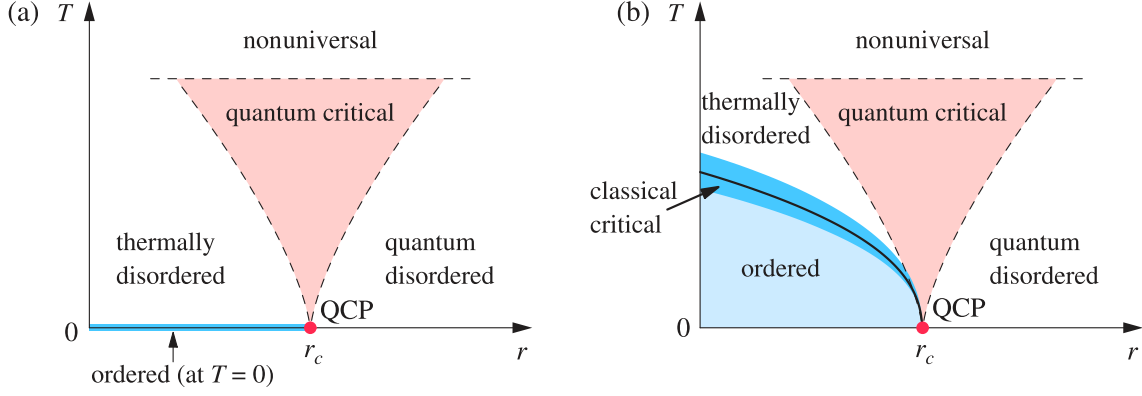
power-law behavior, i.e., the order parameter  $\langle \phi_a \rangle \propto \theta(-t)|t|^\beta$  ( $\langle \phi_a \rangle \propto \text{sgn}(h)|h|^{1/\delta}$ ) with reduced temperature  $t := (T - T_c)/T_c$  (constant external field  $h \equiv J_a(\vec{x})$ ), the specific heat  $C_V = C_\pm |t|^{-\alpha}$ , the order-parameter susceptibility  $\chi = \chi_\pm |t|^{-\gamma}$ , the correlation length  $\xi = \xi_\pm |t|^{-\nu}$ , and the critical two-point correlation function  $G(\vec{x}_1, \vec{x}_2) \propto |\vec{x}_1 - \vec{x}_2|^{-d+2-\eta}$ , with the corresponding mean-field critical exponents  $\alpha = 0$ ,  $\beta = \frac{1}{2}$ ,  $\gamma = 1$ ,  $\delta = 3$ ,  $\nu = \frac{1}{2}$ , and  $\eta = 0$  [56]. Critical exponents are, in general, not independent from each other but are related by Rushbrooke's, Griffiths'/Widom's, and Fisher's scaling laws [56]. Below the upper critical dimension, cf. Sec. 1.1.3, they typically satisfy an additional relation, Josephson's hyperscaling law<sup>2</sup> [46, 56]. Moreover, we would like to emphasize that, in addition to critical exponents, there exist further universal quantities, such as amplitudes ratios ( $C_+/C_-$ ,  $\chi_+/\chi_-$ ,  $\xi_+/\xi_-$ ) or scaling functions [56].

### Quantum phase transitions

Landau-Ginzburg theory can be readily extended to quantum phase transitions, i.e., phase transitions that take place at zero temperature and are tuned by nonthermal parameters, such as pressure, chemical potential, or magnetic field, instead of the temperature [56, 57]. Although absolute zero cannot be reached experimentally, quantum phase transition are still vitally important for experiment, as they structure the phase diagram, cf. Fig. 1.1, also at nonzero temperatures  $T \lesssim a^{-z}$ , where  $a$  denotes a microscopic cutoff scale, such as the lattice spacing, and  $z$  represents the dynamical exponent. This additional universal exponent relates the correlation time  $\tau_c \propto \xi^z$  to the correlation length  $\xi$  and arises, from a formal point of view, through the inclusion of temporal contributions, such as the geometric phase  $\phi_a \partial_\tau \phi_a$  ( $\phi_a \partial_\tau^2 \phi_a$ ) in case of the nonrelativistic (relativistic)  $O(N)$  model [56]. It is worth noting that the relativistic version can be easily rewritten as a classical  $O(N)$  model in  $d + 1$  spatial dimensions. Similarly, the nonrelativistic theory also permits a quantum-to-classical mapping, however, the resulting classical theory in  $d + 1$  dimensions is anisotropic, as the quantum theory possesses only spatial  $O(N)$  symmetry and not full space-time symmetry, a feature already reflected in the tree-level dynamical exponent  $z = 2$ . Although the resulting space is  $(d + 1)$ -dimensional, the nonrelativistic critical theory, in some respects, rather reflects the scaling behavior of a relativistic theory in  $(d + z)$ -dimensional space.

Irrespective of whether a quantum critical point is described by a relativistic or a nonrelativistic quantum field theory, its presence is, from an experimental point of view, most readily revealed through the resulting quantum critical regime, cf. Fig. 1.1, where thermal and quantum fluctuations are of equal importance. This regime is located above (below) the crossover scale  $\propto |r - r_c|^{\nu z}$  ( $\propto a^{-z}$ ), where  $r - r_c$  denotes the renormalized nonthermal tuning parameter, and can be identified by measuring the dynamic structure factor  $\mathcal{S}(\omega, \vec{k})$ , the Fourier-transformed real-time correlator  $G(t_1, \vec{x}_1; t_2 = 0, \vec{x}_2 = 0)$  (assuming

<sup>2</sup>The absence of Josephson's hyperscaling law above the upper critical dimension originates in a nonanalyticity of the Landau-Ginzburg free energy. From a renormalization group point of view, cf. Sec. 1.1.3, the violation is caused by the dangerously irrelevant coupling  $\lambda$  [46, 56].

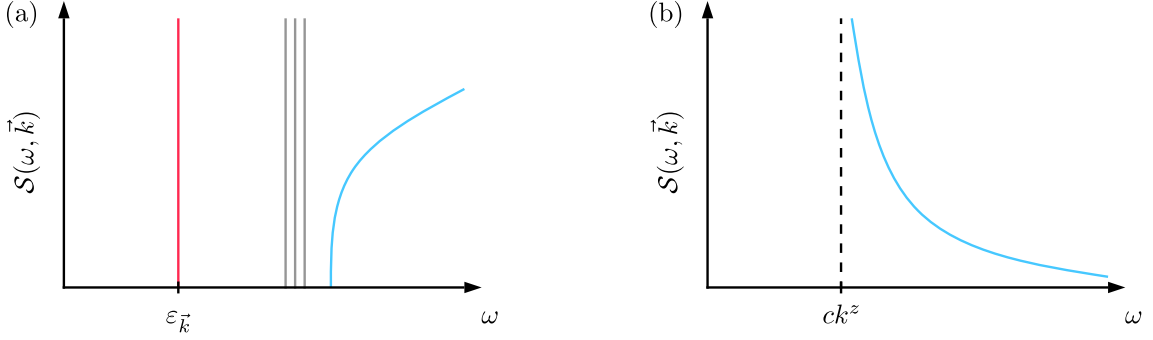


**Figure 1.1.:** Schematic phase diagram as a function of tuning parameter  $r$  and temperature  $T$ , illustrating the influence of a quantum critical point (QCP) even at finite temperature. The strongly interacting quantum critical regime (light-red region), hosting power-law behavior, sets in at the crossover scales  $\propto |r - r_c|^{\nu z}$  and  $\propto a^{-z}$  (dashed lines), where  $a$  denotes a microscopic cutoff scale, such as the lattice spacing. (a) In two spatial dimensions and for sufficiently short-range interactions, the Mermin-Wagner theorem forbids the breaking of continuous symmetries at finite temperature, resulting in a thermally disordered phase at  $T > 0$  and an ordered phase (blue line) only at  $T = 0$ . In case the Mermin-Wagner theorem does not apply, the ordered phase need not exist at  $T > 0$ , in which case the phase diagram remains valid. (b) Alternatively, the ordered phase (light-blue region) may extend to finite temperature, giving rise to a classical critical regime (blue region near solid black transition line). Figure adapted from Ref. [57].

translation invariance in space and time), in neutron or X-ray scattering experiments, for instance [46, 58, 59]. In the ordered phase, as well as in the disordered phase outside the quantum critical regime, the dynamic structure factor displays several characteristic features such as the quasiparticle pole, the quasiparticle continuum, and, generally, poles associated with bound states [46, 58], as depicted in Fig. 1.2(a). In contrast, these features are absent in the strongly interacting quantum critical regime, where there is usually no quasiparticle description available and the dynamic structure factor instead displays a critical continuum [46, 58, 59], as shown in Fig. 1.2(b). Consequently, the quantum critical regime should be understood as a strongly interacting quantum state of matter. Additional experimental signatures of quantum critical behavior include numerous thermodynamic and transport quantities, such as the specific heat, Grüneisen ratio, or conductivity, exhibiting power-law behavior with typically noninteger exponents [46, 58, 59]. We refer the reader to Sec. 2.6 for further discussions of quantum critical signatures and their experimental observation, illustrated with the case of rare-earth pyrochlore iridates.

### 1.1.3. Wilson-Fisher fixed point

Below the upper critical dimension, fluctuations can become important, motivating the search for a framework that systematically incorporates them. This can be achieved by representing the partition function  $Z_{\text{LGW}}$  as a functional integral over field space, i.e. by



**Figure 1.2.:** Schematic graph of the dynamic structure factor  $\mathcal{S}$  as a function of frequency  $\omega$  at fixed momentum  $\vec{k}$ . (a) In both the ordered phase and the disordered phase outside the quantum critical regime, the dynamic structure factor displays a quasiparticle pole (red line) at frequency  $\omega = \varepsilon_{\vec{k}}$ , where  $\varepsilon$  denotes the renormalized dispersion relation. Further characteristic features include the quasiparticle continuum (blue line depicts the three-particle continuum) and, potentially, poles associated with bound states (gray lines). (b) Inside the quantum critical regime, the dynamic structure factor  $\mathcal{S}$  displays a critical continuum (blue line), setting in at frequency  $\omega = ck^z$ , where  $c$  denotes a speed-of-light-like constant and  $z$  the dynamical exponent. Figure inspired by Refs. [46, 58].

“summing up” the weights of all possible order-parameter-field configurations [46, 56],

$$Z_{\text{LGW}} = \int \mathcal{D}\varphi_a \exp(-S), \quad (1.2)$$

where  $S$  denotes the Euclidean action and periodic boundary conditions in imaginary time are implied. This represents the field-theoretical formulation of the Landau-Ginzburg-Wilson paradigm, an extension of Landau-Ginzburg theory that also accounts for *fluctuation effects*, i.e., for field configurations that deviate from the mean-field solution. The latter can be recovered by taking only the stationary field configuration  $\varphi_a^{(\text{MF})} \equiv \phi_a$  into account,  $F_{\text{LG}}[\phi_a] = k_{\text{B}}TS[\varphi_a^{(\text{MF})}]$  [56].

## Renormalization group analysis

Let us return to the thermal  $O(N)$  transition in  $d$  spatial dimensions, or, equivalently, its quantum counterpart in  $d-1$  spatial dimensions, and examine when and how fluctuations lead to deviations from the mean-field results presented in the previous subsection. To this end, we examine the functional integral outlined above, along with the corresponding Euclidean action, the linear sigma model or, equivalently, the  $O(N)$  model, [46, 47]

$$S_{\text{LSM}} = \int d^d x \left[ \frac{1}{2} \varphi_a (r - \nabla^2) \varphi_a + \frac{\lambda}{4!} (\varphi_a^2)^2 \right], \quad (1.3)$$

where  $\varphi_a \equiv \varphi_a(\vec{x})$  denotes the unconstrained real order-parameter field,  $r$  the tuning parameter, and  $\lambda$  the quartic self-interaction<sup>3</sup>. As interacting functional integrals are

<sup>3</sup>Deriving the Landau-Ginzburg theory for the  $O(N)$  transition (Eq. (1.1)) from the linear sigma model (Eq. (1.3)) requires some caution, as the latter involves the rescaled field  $\sqrt{\beta} \varphi_a \mapsto \varphi_a$  and coupling  $\beta^{-1} \lambda \mapsto \lambda$  in order to absorb the inverse temperature  $\beta = (k_{\text{B}}T)^{-1}$  arising from integration over imaginary time.

rarely exactly soluble<sup>4</sup>, we typically rely on perturbative or nonperturbative approximation schemes. In the following, we utilize the  $\epsilon$  expansion around the upper critical dimension  $d_c^+$ , below which fluctuations typically become crucial. This controlled approach in  $\epsilon = d_c^+ - d$  was developed by Wilson and Fisher in the 1970s, based on the idea of deriving an effective theory that captures the physics at large length scales  $L \gg a$ , where  $a$  denotes a microscopic length scale, by successively and systematically integrating out modes with large momenta [56, 70, 71]. The  $\epsilon$  expansion remains a valuable tool to this day, cf. Table 1.1 and Sec. 2.4, and has inspired the development and application of numerous related field-theoretical methods in the study of critical phenomena. In its momentum-shell formulation, one starts by introducing a rescaling factor  $0 \ll b < 1$  to separate low-momentum modes, characterized by  $0 \leq |\vec{k}| < \Lambda/b$ , from high-momentum modes, characterized by  $\Lambda/b \leq |\vec{k}| < \Lambda$ , where  $\vec{k}$  denotes the momentum of a mode and  $\Lambda$  the ultraviolet cutoff. Next, we *eliminate* the high-momentum modes by integrating over the thin momentum shell  $\Lambda/b \leq k < \Lambda$ . Afterwards, we *rescale* both the momenta,  $b\vec{k} \mapsto \vec{k}$ , and the fields,  $b^\Delta \varphi_a(\vec{k}/b) \mapsto \varphi_a(\vec{k})$ , so that the action remains form-invariant. The field renormalization involves an initially unspecified exponent  $\Delta$ , which is typically chosen such that the prefactor of the kinetic term remains unity, yielding  $\Delta = -(d + 2 - \eta)/2$ , where  $\eta$  denotes the anomalous dimension. Carrying out this procedure yields a set of ordinary differential equations for the couplings. At the one-loop order, these renormalization group flow equations take the form<sup>5</sup> [46, 70]

$$\frac{dr}{d \ln b} = 2r + \frac{N+2}{6} \frac{\lambda}{1+r}, \quad (1.4)$$

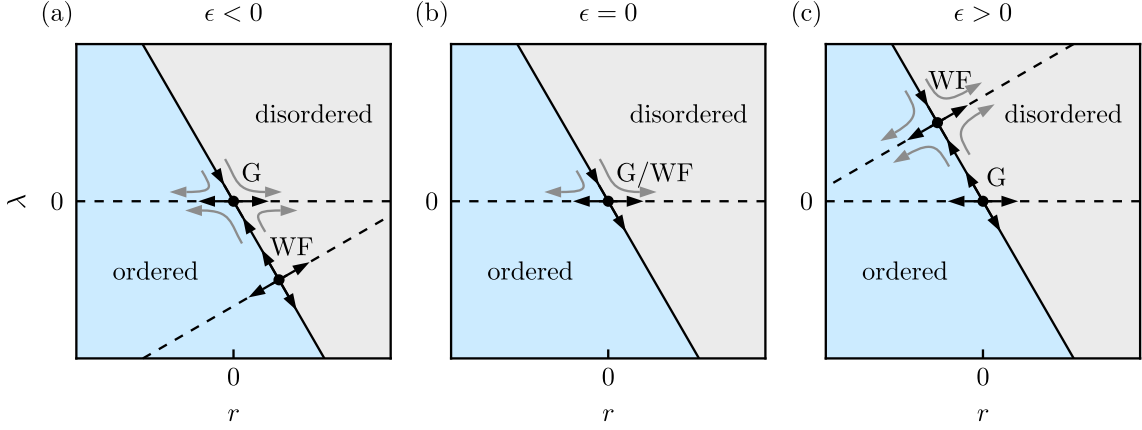
$$\frac{d\lambda}{d \ln b} = (4-d)\lambda - \frac{N+8}{6} \frac{\lambda^2}{(1+r)^2}, \quad (1.5)$$

where we have introduced the dimensionless mass  $r\Lambda^{-2} \mapsto r$  and the dimensionless quartic self-interaction  $\lambda\Lambda^{d-4}S_{d-1}/(2\pi)^d \mapsto \lambda$ , and  $S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$  denotes the surface area of the unit  $(d-1)$ -sphere. At the one-loop order, the anomalous dimension  $\eta$  vanishes and thus does not contribute to the beta functions (right-hand sides of the flow equations) [46, 75].

The above flow equations possess two stationary points, the Gaussian fixed point at  $r_\star = \lambda_\star = 0$  and the Wilson-Fisher fixed point at  $r_\star = -(N+2)/(2(N+8))\epsilon + \mathcal{O}(\epsilon^2)$  and  $\lambda_\star = 6/(N+8)\epsilon + \mathcal{O}(\epsilon^2)$ , which undergo a transcritical bifurcation as a function of the dimension  $d = d_c^+ - \epsilon$  of great formal and physical importance [46, 56, 70]. Let us unpack this statement by considering the following three cases:  $\epsilon < 0$ ,  $\epsilon = 0$ , and  $\epsilon > 0$ . Above the upper critical dimension  $d_c^+ = 4$ , i.e.,  $\epsilon < 0$ , the Gaussian fixed point features one relevant

<sup>4</sup>Prominent counterexamples, i.e., interacting field theories which are indeed exactly soluble, are numerous fermionic theories in  $D = 1 + 1$  (which can be mapped to noninteracting bosonic theories) [60–64], conformal field theories in  $D = 1 + 1$  (as a consequence of the constraining Virasoro algebra) [65, 66], or the Sachdev-Ye-Kitaev model in  $D = 0 + 1$  (whose exactness relies on the use of disorder averages) [67–69]. For more information on the latter, see Sec. 4.

<sup>5</sup>Being exact in  $N$ , it is tempting to extrapolate the flow equations other values  $N \notin \mathbb{N}_{>0}$ . One might therefore wonder if one can assign a meaning to such cases, which turns out to be true, at least for  $N = 0$  and  $N = -2$  corresponding to self-avoiding random walks [72] and loop-erased random walks [73, 74], respectively.



**Figure 1.3.:** Schematic renormalization group flow of the linear sigma model as a function of the order-parameter mass  $r$  and the quartic self-interaction  $\lambda$ , for different values of the parameter  $\epsilon = 4 - d$ , where  $d$  denotes the dimension. The displayed flow topology is valid for any number of order-parameter components  $N > 0$ . Arrows point towards the infrared and the disordered (ordered) phase is indicated in gray (blue). Separatrices (other eigenspaces) are depicted as solid (dashed) black lines. (a) Above the upper critical dimension,  $\epsilon < 0$ , the Gaussian fixed point (G) is critical and governs the  $O(N)$  transition, while the Wilson-Fisher fixed point (WF) is unstable. (b) At  $\epsilon = 0$ , the two fixed points coincide. (c) Below the upper critical dimension,  $\epsilon > 0$ , the Gaussian fixed point is unstable, while the Wilson-Fisher fixed point is critical and mediates the transition. Figure inspired by Ref. [58].

and one irrelevant direction, as shown in Fig. 1.3(a), and thus serves as the critical fixed point governing the  $O(N)$  transition, leading to the noninteracting mean-field exponents discussed in the previous subsection [56, 70]. In this scenario, the Wilson-Fisher fixed point possesses two relevant directions and is therefore an unstable fixed point. At the upper critical dimension,  $\epsilon = 0$ , the Gaussian and Wilson-Fisher fixed points coincide, as illustrated in Fig. 1.3(b), resulting in corrections to scaling in the form of multiplicative logarithms with fractional powers<sup>6</sup> [56, 76, 77]. Below the upper critical dimension,  $\epsilon > 0$ , the two fixed points have exchanged their stability (mainly) associated with the quartic self-interaction, see Fig. 1.3(c). As a result, the Gaussian fixed point is fully unstable and no longer describes the  $O(N)$  transition. Instead, the phase transition is now governed by the Wilson-Fisher fixed point, giving rise to interacting critical exponents that differ from those of mean-field theory [46, 56, 70]. This becomes apparent, for instance, by determining the correlation-length exponent  $\nu$  through a linear stability analysis of the flow equations near the Wilson-Fisher fixed point, i.e., by computing the inverse of the only positive eigenvalue of the stability matrix  $M_{ij} := \partial_j \beta_i|_{\text{WF}}$ , where  $i$  ( $j$ ) labels the beta functions (couplings) [56]. The correlation-length exponent takes the form  $\nu = 1/2 + (N + 2)/(4(N + 8)\epsilon + \mathcal{O}(\epsilon^2))$ , a result that clearly deviates from the mean-field value [47, 56]. As briefly mentioned before, the one-loop renormalization group analysis results in a trivial anomalous dimension  $\eta = 0$ . In order to see the interacting nature of this critical exponent as well, one has to incorporate the two-loop “sunrise” diagram, for instance, via field-theoretical renormalization, yielding indeed a nontrivial  $\eta = (N + 2)/(2(N +$

<sup>6</sup>Similar phenomenology can also be encountered below the upper critical dimension. We refer the reader to Chap. 2, where the influence of marginally irrelevant operators at critical fixed points is discussed in detail. Logarithmically slow flows can also arise from saddle-node bifurcations, see Sec. 1.3.2, Sec. 1.4.3, or Chap. 3.

$8)^2\epsilon^2 + \mathcal{O}(\epsilon^3)$  [46, 56, 75]. All other critical exponents can be readily obtained from the scaling relations mentioned in the previous subsection (including Josephson’s hyperscaling relation). Note that the stability matrix  $M$  also possesses a negative eigenvalue and its absolute value sets the corrections-to-scaling exponent  $\omega = \epsilon + \mathcal{O}(\epsilon^2)$  [47, 56]. In general, the corrections-to-scaling exponent is defined as the absolute value of the least negative<sup>7</sup> eigenvalue of the stability matrix and characterizes how quickly perturbations on the critical manifold decay under the renormalization group flow.

We would like to conclude the renormalization group analysis of the  $O(N)$  transition with a few remarks. First, the existence a critical fixed point, both below and above the upper critical dimension, substantiates the assumption of universality made in the previous subsection. Second, higher-order terms not included in the linear sigma model, Eq. (1.3), such as a contribution  $\propto \nabla^2(\varphi_a^2)^2$ , can be safely neglected, as they are power-counting irrelevant in  $d > 2$  dimensions [56, 58]. Third, taking into account higher-loop corrections typically improves the estimates for the critical exponents, but requires the use of resummation methods, since the  $\epsilon$  expansion yields only an asymptotic series [78]. The correlation-length exponent  $\nu$ , the anomalous dimension  $\eta$ , and the corrections-to-scaling exponent  $\omega$  of the state-of-the-art seven-loop order  $4 - \epsilon$  expansion are, for comparison, included in Table 1.1 and discussed in the next paragraph. Further, since one of the main conclusions of this subsection is the significance of fluctuations below the upper critical dimension  $d_c^+$ , we would like to point out that the dimension may even be reduced to the lower critical dimension  $d_c^- < d_c^+$  and below, where fluctuations are sufficiently strong to prevent the formation of long-range order. Remarkably, critical phenomena can still occur at and below the lower critical dimension, see Sec. 1.2.1, for instance. Lastly, from a formal perspective, a central objective of all research chapters in this dissertation is to identify and analyze strongly coupled critical fixed points similar to, but generally more involved than, the Wilson-Fisher fixed point encountered in this subsection.

## Complementary approaches

The functional integral, Eq. (1.2), is typically amenable to a variety of methods. In the case of the  $O(N)$  transition, we would like to point out prominent approaches such as the perturbative renormalization group in fixed dimension [78], the high-temperature expansion [92], and classical Monte Carlo simulations [93]. These methods serve as complementary approaches to the mean-field analysis and the  $4 - \epsilon$  expansion discussed above. Less conventional, but equally valuable, are methods such as the nonperturbative functional renormalization group framework [94] and the controlled large- $N$  critical point formalism [95]. We refer the reader to Chap. 3 and Chap. 4 for more information on the functional renormalization group and the large- $N$  expansion, respectively. In case the

<sup>7</sup>This definition includes the case of a vanishing corrections-to-scaling exponent,  $\omega = 0$ , occurring whenever a marginal operator is present. For instance, this is the case at every fixed-point collision, and thus explains  $\omega = 0$  right at the upper critical dimension. Further, instances of vanishing corrections-to-scaling exponents are noted in the previous footnote or, more explicitly, listed in Table 3.1.

**Table 1.1.:** Correlation-length exponent  $\nu$ , anomalous dimension  $\eta$ , and corrections-to-scaling exponent  $\omega$  of the Ising universality class ( $N = 1$ ) in  $d = 3$  spatial dimensions determined by different methods. High-precision results are presented along with their uncertainties. In both the conformal bootstrap and the fuzzy sphere regularization frameworks, the exponents  $\nu$ ,  $\eta$ , and  $\omega$  can be readily extracted from the scaling dimensions  $\Delta_\epsilon = d - 1/\nu$ ,  $\Delta_\sigma = (d - 2 + \eta)/2$ , and  $\Delta_{\epsilon'} = d + \omega$  of the corresponding primary operators [79].

| Method                                     | $\nu$          | $\eta$          | $\omega$    |
|--------------------------------------------|----------------|-----------------|-------------|
| Mean-field theory [56]                     | 0.5            | 0               | —           |
| 4 - $\epsilon$ expansion (one loop) [70]   | 0.58 $\bar{3}$ | 0               | 1           |
| 4 - $\epsilon$ expansion (seven loop) [80] | 0.62977(22)    | 0.03653(65)     | 0.82311(50) |
| Perturbative renormalization group [81]    | 0.6304(13)     | 0.0335(25)      | 0.799(11)   |
| High-temperature expansion [82]            | 0.63012(16)    | 0.03639(15)     | 0.83(5)     |
| Classical Monte Carlo [83, 84]             | 0.62998(5)     | 0.036284(40)    | 0.832(6)    |
| Functional renormalization group [85, 86]  | 0.63012(16)    | 0.0361(11)      | 0.832(14)   |
| Conformal bootstrap [87, 88]               | 0.62997097(12) | 0.036297612(48) | 0.82968(23) |
| Fuzzy sphere regularization [89, 90]       | 0.6303         | 0.05            | 0.838       |
| Fluid mixture experiment [91]              | 0.629(3)       | 0.032(13)       | —           |

transition of interest is captured by an emergent conformal field theory<sup>8</sup>, as conjectured for the three-dimensional Ising model [98, 99], further methods such as the conformal bootstrap [97] or the newly developed<sup>9</sup> fuzzy sphere regularization [89] may become applicable. Table 1.1 presents the correlation-length exponent  $\nu$ , the anomalous dimension  $\eta$ , and the corrections-to-scaling exponent  $\omega$  of the Ising universality class ( $N = 1$ ) in  $d = 3$  spatial dimensions determined by the methods pointed out above<sup>10</sup>, highlighting the excellent agreement among high-precision results. Additionally, we include the correlation-length exponent  $\nu$  and the anomalous dimension  $\eta$  obtained from a fluid mixture experiment [91], showing also good agreement between theory and experiment.

## 1.2. Phase transitions beyond conventional symmetry breaking

Having reviewed the Landau-Ginzburg-Wilson paradigm in the previous section, we now turn to its limitations. A first route beyond this framework is constituted by phase transitions that evade a description in terms of an action for a local order-parameter field. In this section, we explore two classes of such phase transitions, both sometimes termed topological phase transitions, despite involving concepts of topology in rather different ways.

<sup>8</sup>A conformal field theory in  $D > 2$  space-time dimensions constitutes a field theory which is invariant under Poincaré transformations, scale transformations, and special conformal transformations, i.e., possesses  $\text{SO}(D+1, 1)$  ( $\text{SO}(D, 2)$ ) symmetry in Euclidean (Lorentzian) signature [96, 97]. All conformal field theories encountered in this dissertation belong to this class and are less restricted than their highly symmetric two-dimensional analogues.

<sup>9</sup>We note that the fuzzy sphere regularization does not, at present, yield the most accurate critical exponents. Our purpose in mentioning it here is rather to showcase that new methods, with their own advantages and disadvantages compared to established approaches, are still being actively developed to address long-standing problems.

<sup>10</sup>We note that although various loop corrections have been computed within the large- $N$  critical point formalism [100–102], we are not aware of any (successful) resummation results (in the spirit of Refs. [103, 104]) for the Ising universality class in  $d = 3$  spatial dimensions that could have been included in Table 1.1. We thank John A. Gracey for helpful discussions on this point.

### 1.2.1. Berezinskii-Kosterlitz-Thouless transition

To illustrate the first class, consider the two-dimensional XY-model, described by the Hamiltonian [105]

$$H_{\text{XY}} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j, \quad (1.6)$$

where  $\vec{S}_i$  denotes a classical spin of unit length at site  $i$ , the sum runs over nearest neighbors on the square lattice, and  $J > 0$  represents a ferromagnetic coupling, without loss of generality. As in most physical systems<sup>11</sup>, the high-temperature phase here is disordered and, to leading order, characterized by spin-spin correlations that decay exponentially with distance  $|\vec{x}|$  [105],

$$\langle \vec{S}(\vec{x}) \cdot \vec{S}(\vec{0}) \rangle \propto \exp\left(-\frac{|\vec{x}|}{\xi}\right), \quad (1.7)$$

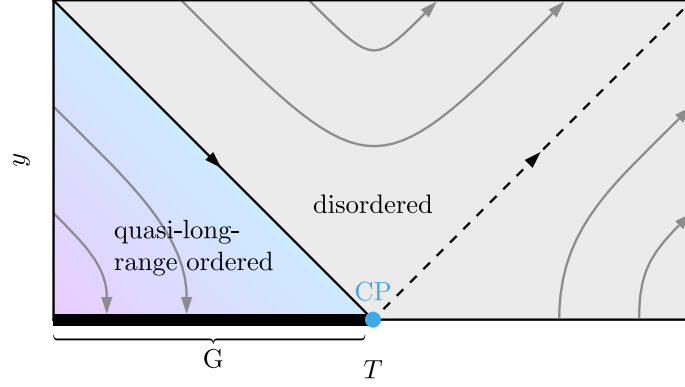
where  $\xi$  denotes the correlation length. One might be tempted to conclude that the disordered phase extends down to arbitrarily small temperatures  $T > 0$ , since long-range order, associated with U(1) symmetry breaking, is forbidden at finite temperature by the Mermin-Wagner theorem [110, 111]. To our surprise, the model does undergo a transition at  $(T/J)_c = 0.893\dots$  [112–114] into a phase with quasi-long-range order characterized by algebraically decaying correlations [14, 15],

$$\langle \vec{S}(\vec{x}) \cdot \vec{S}(\vec{0}) \rangle \propto \frac{1}{|\vec{x}|^\eta}, \quad (1.8)$$

with anomalous dimension  $\eta > 0$ . In agreement with the Mermin-Wagner theorem, the Berezinskii-Kosterlitz-Thouless transition is not to be understood in terms of spontaneous symmetry breaking<sup>12</sup>, but rather as symmetry-preserving deconfinement transition of vortex-antivortex pairs [14, 15, 117–119]. The underlying mechanism can already be understood qualitatively within a single-vortex picture, i.e., by analyzing the free energy  $F_v = E_v - TS_v$  of an individual vortex. In the low-temperature phase, the energy cost  $E_v$  for a vortex outweighs the entropic contribution  $TS_v$ , and vortex-antivortex pairs remain predominantly bound. In contrast, in the high-temperature phase, the entropic contribution dominates, leading to a proliferation of vortices. Given the duality between the two-dimensional XY-model and the two-dimensional lattice Coulomb gas, the low-temperature phase can alternatively be understood as a dipole phase enforced by the long-range Coulomb potential, while the high-temperature phase corresponds to a phase of relatively free charges due to the screened Coulomb interaction [105, 120, 121]. As in the XY-model, the phase transition is driven by topological defects, in this case, quantized charges.

<sup>11</sup>Exceptions to this rule, namely systems that remain ordered up to arbitrarily high temperatures, are discussed in the series of works Refs. [106–109].

<sup>12</sup>Note that long-range analogues of the XY-model exhibit an even richer phase diagram, including long-range-ordered phases, see Refs. [115, 116], for instance.



**Figure 1.4.:** Schematic renormalization group flow diagram of the sine-Gordon model as a function of the temperature  $T$  and the fugacity  $y$ . Arrows point towards the infrared. The quasi-long-range-ordered phase (blue-to-purple gradient) is controlled by an attractive line of stable Gaussian fixed points ( $G$ ). The color gradient indicates that each trajectory flows to a different ultraviolet fixed point, resulting in a temperature-dependent anomalous dimension. The Berezinskii-Kosterlitz-Thouless transition into the disordered phase (gray) is governed by the rightmost Gaussian fixed point, the critical fixed point ( $CP$ ). The corresponding separatrix and the other eigendirection are depicted as solid and dashed black lines, respectively.

Both theories share their universality class with a third model, the two-dimensional sine-Gordon model, whose continuum formulation is given by the Euclidean action [56, 78]

$$S_{SG} = \int d^2x \left[ \frac{T}{2} (\partial_\mu \phi)^2 - 2y \cos(2\pi\phi) \right], \quad (1.9)$$

where  $\phi$  denotes an unconstrained real scalar field,  $T$  the dimensionless temperature in units of  $J = 1$ , and  $y$  the effective fugacity in the continuum. A (soft-cutoff) momentum-shell renormalization group analysis to leading order in the fugacity reveals the flow equations [56, 119]

$$\frac{dT}{d \ln b} = \frac{y^2}{2T}, \quad (1.10)$$

$$\frac{dy}{d \ln b} = \left( 2 - \frac{\pi}{T} \right) y, \quad (1.11)$$

with dimensionless fugacity  $(4\pi)^2 y / \Lambda^2 \mapsto y$ , ultraviolet cutoff  $\Lambda$ , and renormalization group scale  $\ln b$ . As visualized in Fig. 1.4, the Berezinskii-Kosterlitz-Thouless phase is the result of an attractive line of Gaussian fixed points at fugacity  $y_\star = 0$  and temperatures  $0 \leq T_\star < \pi/2$ . Each trajectory within this quasi-long-range-ordered phase terminates at a distinct ultraviolet fixed point, resulting in a temperature-dependent anomalous dimension  $\eta$ . As the fugacity  $y$  is irrelevant in the low-temperature phase, the sine-Gordon model exhibits wave-like behavior, as known from massless Klein-Gordon theory [47]. In contrast, the high-temperature phase corresponds to an ordinary disordered fixed point, as indicated by the runaway flow towards  $y \rightarrow \infty$  and  $T \rightarrow \infty$ . Here, the fugacity  $y$  is relevant and the sine-Gordon model exhibits pinning behavior, with constant field configurations  $\phi(x) \approx z$ , with  $z \in \mathbb{Z}$ , dominating, where different minima of the action are connected by the creation and annihilation of kink-antikink pairs [122]. The Berezinskii-Kosterlitz-Thouless transition is mediated by the Gaussian critical fixed point at fugacity  $y_\star = 0$  and

temperature  $T_\star = \pi/2$ . Its noninteracting nature justifies the intuitive single-vortex-free-energy argument presented above and underpins our understanding of this transition as an unbinding transition of vortex-antivortex pairs [56].

We conclude that the Berezinskii-Kosterlitz-Thouless transition lies beyond the Landau-Ginzburg-Wilson paradigm, as it is not characterized by spontaneous symmetry breaking<sup>13</sup> but instead signaled by nonlocal measures, such as vortex condensation. Lastly, we emphasize that the ideas discussed in this subsection (1) are relevant for a variety of systems, such as superconducting thin films, and (2) have been experimentally confirmed [124].

### 1.2.2. Phase transitions from topological order

A second notion of topological phase transitions concerns phase transitions either between trivial and topologically ordered phases or even between phases with distinct topological order [125–128]. From a formal perspective, topologically ordered phases are usually<sup>14</sup> characterized by topological field theories in the infrared, i.e., field theories for which all correlation functions exhibit diffeomorphism invariance [130–135]. A historic example of such a topological field theory is the  $U(1)_k$  Chern-Simons theory [125, 132, 136, 137],

$$S_{\text{CS}} = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma, \quad (1.12)$$

the low-energy effective field theory for the fractional quantum Hall effect with Laughlin-type filling fraction  $\nu = 1/k$ , where  $k \in 2\mathbb{N}_{>0} + 1$ . It can be shown that the emergent gauge field  $a$  attaches a magnetic flux of  $2\pi/k$  to electronic quasiparticles in the presence of an external magnetic field, giving rise to Abelian braiding statistics [135, 138, 139]. As in the previous subsection, and most importantly for our discussion here, fractional quantum Hall phases, and phases with topological order in general, cannot be characterized by local order parameters but require different nonlocal probes. In two spatial dimensions<sup>15</sup>, such topological diagnostics are the ground state degeneracy induced by the nontrivial topology of the underlying (closed) space [125–127] and the geometric phase acquired when deforming this space via modular transformations [126, 142, 143].

Equivalently, and again in two spatial dimensions<sup>16</sup>, topologically ordered phases can be identified directly through their long-range entanglement, which manifests in an unconventional universal contribution to the von Neumann entropy [147–149],

$$\mathcal{S} = \alpha \ell - \gamma + \mathcal{O}(\ell^{-1}), \quad (1.13)$$

<sup>13</sup>For an alternative characterization of the Berezinskii-Kosterlitz-Thouless transition via higher-form symmetries, cf. Sec. 1.2.2, and their anomalies, see Ref. [123].

<sup>14</sup>For topological orders that evade a low-energy description by a topological quantum field theory, see, for instance, Ref. [129].

<sup>15</sup>For a classification of topological orders in three spatial dimension with only point-like excitations, see Refs. [140, 141].

<sup>16</sup>For a discussion of the entanglement entropy’s rich structure in three spatial dimensions and higher see, for example, Refs. [144–146].

measuring the entanglement between a large simply connected region, where  $\ell$  denotes the length of the smooth<sup>17</sup> boundary, and its complement. In contrast to the perimeter-law contribution, that comes with a nonuniversal constant  $\alpha$  [154, 155], the topological entanglement entropy  $-\gamma = -\ln \mathcal{D}$  is universal and determined by the total quantum dimension  $\mathcal{D} \geq 1$  of the system, where  $\mathcal{D} = 1$  ( $\mathcal{D} > 1$ ) corresponds to trivial (nontrivial) topological order. For the fractional quantum Hall systems discussed above, the quantum dimension simply reads  $\mathcal{D} = \sqrt{k}$ , which highlights the deep connection between the topological entanglement entropy and the topological quantum field theory of a given system, in this case, via the level  $k$  [147].

A third characterization of topological orders returns to the concept of spontaneous symmetry breaking. Although topologically ordered phases cannot be understood from the viewpoint of spontaneous symmetry breaking of ordinary global (0-form) symmetries, it is possible to understand Abelian topological orders (characterized by trivial fusion algebras with group structure) as phases of spontaneously broken  $p$ -form symmetries, with  $p \in \mathbb{N}_{>0}$  [55, 156, 157]. In this way, the fractional quantum Hall phase with a Laughlin-type filling fraction  $1/k$  can be viewed as a phase with spontaneously broken  $\mathbb{Z}_k$  1-form symmetry, diagnosed by perimeter-law behavior of the charged loop operator [55, 157]. Although non-Abelian topological orders (defined by nontrivial fusion algebras) cannot be characterized by broken higher-form symmetries, they can still be understood as phases of spontaneously broken symmetries through a further generalization of the concept of symmetry, namely categorical symmetries [157–159]. Since generalized symmetries, such as higher-order symmetries and categorical symmetries, can quickly become a rather technical subject and are not the main focus of this dissertation, we refer the reader, for instance, to Refs. [55, 157, 160] for further background. One might be tempted to conclude that extending the Landau-Ginzburg-Wilson paradigm to also include spontaneous symmetry breaking of generalized symmetries yields a framework that describes many phases of matter precisely in the spirit of Landau. However, we would like to emphasize that the conventional Landau-Ginzburg-Wilson paradigm, as presented in Sec. 1.1, is built on symmetries present in the microscopic model, whereas the generalized symmetries encountered in this paragraph are emergent in the infrared and absent at the microscopic level, thus conceptually distinct [161].

We want to conclude this subsection on phase transitions from topological order with a few remarks. First, topologically ordered phases should not be conflated with symmetry-protected topological phases because the latter are “only” short-range entangled phases with trivial topological order in the bulk [162–164]. Examples of symmetry-protected topological phases, i.e., phases that cannot be connected to the trivial phase without breaking a global symmetry, include the Haldane phase of the spin-1 chain [165–168] and topological band insulators [169–171]. However, the boundary of a symmetry-protected topological bulk phase may realize surface topological order, provided it preserves all global symmetries and is gapped [172]. Phase transitions involving such surfaces lie beyond the

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<sup>17</sup>For a discussion of corner contributions see Refs. [150–153], for instance.

Landau-Ginzburg-Wilson paradigm, in the manner outlined in this subsection. Second, note that there also exist other phases featuring long-range entanglement, such as fracton phases [173] or gapless quantum-spin-liquid phases [174]. We will return to the latter in the context of kagome quantum magnets in Sec. 3.6. Lastly, we emphasize that there are various other notions of topological phase transitions beyond those outlined in this section. One such example will be encountered in Chap. 2, in the form of a Lifshitz transition between a single quadratic band touching point and four pairs of Weyl cones.

### 1.3. Phase transitions enriched by additional gapless modes

A second route beyond the Landau-Ginzburg-Wilson paradigm is constituted by phase transitions that cannot be characterized in terms of local and analytic order-parameter theories. In this section, we discuss two realizations of such phase transitions induced by the presence of additional gapless degrees of freedom, namely gapless fermions and gauge bosons.

#### 1.3.1. Breakdown of Hertz-Millis theory

Given the tremendous success of the Landau-Ginzburg-Wilson paradigm, it is tempting to reformulate problems involving fermionic degrees of freedom as purely bosonic order-parameter theories. A standard way to pursue this strategy is to partially bosonize an entirely fermionic theory via a Hubbard-Stratonovich transformation in one or several channels, and subsequently integrate out the fermion fields, resulting in a formally equivalent theory for one or more order parameters alone [175, 176]. This was precisely the strategy that Hertz initially, and later Millis and others, attempted to follow in the context of two-dimensional metals [177–180]. For a metal close to an Ising nematic instability [46, 181–184], for instance, the resulting effective action takes the form [182, 183, 185, 186]

$$S_{\text{HM}} = \sum_{n=1}^{\infty} \int_{\omega_1, \vec{k}_1} \dots \int_{\omega_{2n}, \vec{k}_{2n}} \Gamma^{(2n)}(\omega_1, \vec{k}_1, \dots, \omega_{2n}, \vec{k}_{2n}) \phi(\omega_1, \vec{k}_1) \dots \phi(\omega_{2n}, \vec{k}_{2n}), \quad (1.14)$$

where  $\phi$  denotes the Ising order-parameter field and  $\int_{\omega, \vec{k}} := \int \frac{d\omega d^2k}{(2\pi)^3}$ . While correctly accounting for Landau damping<sup>18</sup> in the two-point contribution  $\Gamma^{(2)}$ , implying a dynamical exponent  $z = 3$  at tree level, Hertz assumed the four-point vertex function  $\Gamma^{(4)}$  to be, at leading order, frequency- and momentum-independent (apart from the usual terms enforcing frequency and momentum conservation). Further, he assumed that it is permissible to truncate the infinite series of collective modes after the fourth-order term, essentially assuming all contributions  $\Gamma^{(2n)}$ ,  $n \geq 3$ , to be irrelevant. Via a power counting argument

<sup>18</sup>Note that integrating over all momenta and frequencies, and not only over a thin momentum and/or frequency shell, allows nonanalyticities to arise. This contrasts with momentum-shell renormalization group treatments, which yield only analytic contributions.

he arrived at the conclusion that the quartic self-interaction is irrelevant in spatial dimensions  $d > 1$ , as the system is effectively  $(d + z)$ -dimensional, and the Gaussian fixed point must govern the critical behavior, cf. Sec. 1.1.3.

As realized decades later, these assumptions can break down dramatically in  $d = 2$  spatial dimensions, due to the intricate frequency and momentum dependence of vertex functions [185, 187, 188]. In particular, nonanalyticities of higher-order vertex functions  $\Gamma^{(2n \geq 4)}$  can drastically change the power counting and even lead to marginal terms at every order in the expansion [188]. Such nonanalyticities correspond to nonlocalities in imaginary time and real space, and together with a potentially non-truncatable series, clearly go beyond the Landau-Ginzburg-Wilson paradigm. It is worth noting that criticality beyond Hertz-Millis theory is not only of theoretical interest but also of broad experimental relevance, for instance, in the context of cuprates [189–193], iron-based materials [194–198], or heavy-fermion compounds [199].

In summary, although integrating out the gapless fermionic degrees of freedom results in formally equivalent order-parameter theories, such theories might be of very limited practical utility due to their great complexity. In such cases, it is usually far more convenient to treat bosons and fermions on equal footing by working within a partially bosonized framework, or even within a purely fermionic formulation. For a comprehensive overview of methods applicable to the model discussed above and its close relatives, see Refs. [46, 200–202], for example.

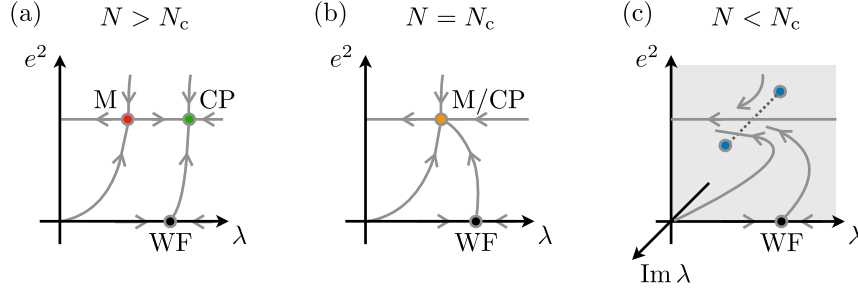
### 1.3.2. Fluctuation-induced first-order phase transitions

The previous subsection raises the question of how important particle statistics are in this context, and whether additional gapless bosonic modes can similarly give rise to order-parameter theories that go beyond the Landau-Ginzburg-Wilson paradigm. To address this questions, we turn once again to a specific example, the Abelian Higgs model, characterized by the Euclidean action [44, 56, 203]

$$S_{\text{AH}} = \int d^3x \left[ |D_\mu \phi|^2 + \frac{1}{4} F_{\mu\nu}^2 + r|\phi|^2 + \lambda(|\phi|^2)^2 \right], \quad (1.15)$$

where  $\phi$  denotes an  $N$ -component complex field with mass  $r$  and quartic self-interaction  $\lambda$ . This bosonic field is coupled to a noncompact  $U(1)$  gauge field  $A_\mu$ , incorporated in the field-strength tensor  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , via the covariant derivative  $D_\mu = \partial_\mu - ieA_\mu$  and the charge  $e$ . For  $N = 1$ , the Abelian Higgs model describes the phase transition from a normal metal to a superconductor [44, 56, 204, 205], as well as the transition from a nematic phase to a smectic- $A$  phase in the context of liquid crystals [56, 204, 206, 207]. Further, the above model constitutes a prime example displaying the Anderson-Higgs mechanism [208–211], responsible for mass generation of the photon in superconductors, and the  $W^\pm$  and  $Z^0$  bosons of the standard model<sup>19</sup>. Moreover, the Abelian Higgs model

<sup>19</sup>As this introductory chapter primarily focuses on concepts, one might wonder if gauge fermions can also acquire a mass via an Anderson-Higgs-like mechanism, which turns out to be possible in certain



**Figure 1.5.:** Schematic fixed-point structure of the  $N$ -component Abelian Higgs model in the parameter space spanned by the quartic self-interaction  $\lambda$  and the charge  $e^2$ . (a) For  $N > N_c$ , the flow equations possess three nontrivial fixed points, the uncharged Wilson-Fisher fixed point (WF), the charged critical fixed point (CP), and the charged multicritical point (M). (b) At  $N = N_c$ , the fixed points CP and M collide, and (c) for  $N < N_c$ , they have annihilated, i.e., they are complex fixed points. The annihilation leaves behind a runaway flow, indicating the fluctuation-induced first-order phase transition. Figure adapted from Ref. [215].

plays a central role in the description and interpretation of deconfined quantum critical points, cf. Sec. 1.4.

In the limit of a large number of order-parameter components,  $N \rightarrow \infty$ , order-parameter fluctuations are suppressed and mean-field theory becomes exact, cf. Sec. 2.3 and Sec. 3.4. This approach, applicable to type-I superconductors for instance, allows us to integrate out the gauge field in a straightforward manner. Remarkably, in addition to the analytic mass and quartic self-interaction contribution, the gauge field induces a nonanalytic term in field space, a contribution  $\propto |\phi|^3$  with negative prefactor, which renders the phase transition first order at the mean-field level [56, 204]. This proves that additional gapless bosonic modes, such as gauge fields, can also lead to nonanalytic order-parameter theories that lie beyond the Landau-Ginzburg-Wilson paradigm.

It is natural to wonder whether the situation changes in the presence of strong order-parameter fluctuations, relevant to type-II superconductors or deconfined quantum critical points, for example [56, 204, 215]. According to our current knowledge, a one-loop  $4 - \epsilon$  expansion, accounting for both order-parameter and gauge-field fluctuations, already reveals the correct qualitative behavior [215]. At criticality,  $r = 0$ , the resulting flow equation read [215, 216]

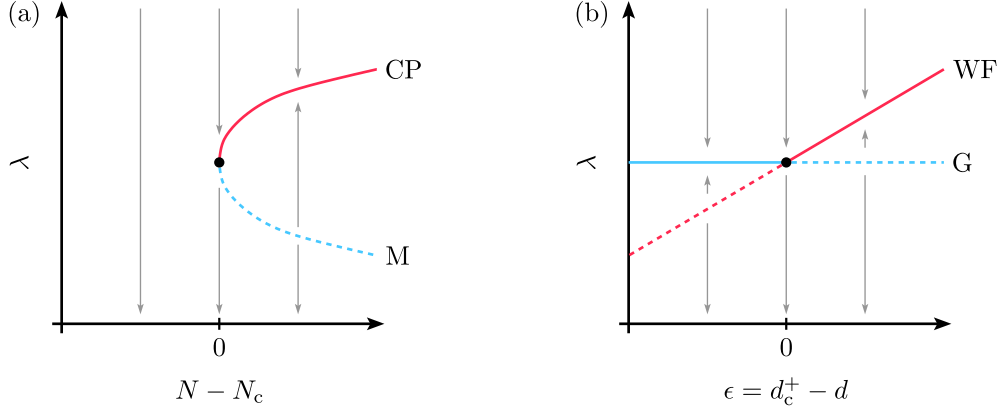
$$\frac{de^2}{d \ln b} = (4 - d)e^2 - \frac{N}{3}e^4, \quad (1.16)$$

$$\frac{d\lambda}{d \ln b} = (4 - d)\lambda - 6e^4 + 6e^2\lambda - (N + 4)\lambda^2, \quad (1.17)$$

with rescaled couplings  $(e^2, \lambda)\Lambda^{d-4}S_{d-1}/(2\pi)^d \mapsto (e^2, \lambda)$  and space-time dimension  $d$ . Apart from the Gaussian and the Wilson-Fisher fixed point at  $e^2 = 0$ , which we have already encountered in Sec. 1.1, the flow equations host two additional fixed points at  $e^2 > 0$  for a large enough number of complex components  $N > N_c = 182.951 \dots$  [215,

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supergravity theories, where the spin- $\frac{3}{2}$  gravitino field is able to “eat up” the spin- $\frac{1}{2}$  goldstino field and becomes massive [212–214].



**Figure 1.6.:** Comparison of the two primary fixed-point bifurcations discussed in this dissertation. Stable (unstable) fixed points are depicted as solid (dashed) lines.  $\lambda$  denotes a coupling constant, such as the quartic self-interaction. (a) Saddle-node bifurcation as realized in the Abelian Higgs model. Increasing the order-parameter components  $N$  beyond a critical number  $N_c$  yields a critical fixed point (CP) and a multicritical fixed point (M). (b) Transcritical bifurcation as encountered for instance in the linear sigma model, cf. Sec. 1.1.3. Lowering the dimension  $d = d_c^+ - \epsilon$  below the upper critical dimension  $d_c^+$  leads to critical behavior no longer governed by the Gaussian fixed point (G), but instead by the Wilson-Fisher fixed point (WF). Figure adapted from Ref. [217].

[216]. The charged fixed points are identified as a multicritical fixed point and a critical fixed point, the latter governing the metal-to-superconductor transition [215, 216]. A schematic representation of the fixed-point structure is shown in Fig. 1.5. Lowering the number of components to  $N = N_c$  leads to a collision of the critical and multicritical fixed points. Upon further decreasing  $N$ , both fixed points annihilate, i.e., they become complex fixed-point solutions of the flow equations [56, 215, 216]. This marks the onset of the fluctuation-induced first-order phase transition. More accurate estimates might be obtained by employing higher-loop calculations in combination with resummation and interpolation methods, with  $N_c(d = 3) = 12.2 \pm 3.9$  in case of the state-of-the-art four-loop calculation [215].

Let us end this subsection with a few remarks. First of all, fluctuation-induced first-order transitions have not only been of theoretical interest but have also remained relevant across various experimental fields over time [218–220]. Second, the reverse scenario of a “fluctuation-induced second-order transition”, i.e., a discontinuous phase transition turning continuous upon coupling the order-parameter theory to additional gapless modes, is also possible. When realized through additional fermionic degrees of freedom, these critical points are known as fermion-induced quantum critical points [221–224]. Moreover, a similar nonanalyticity  $\propto |\phi|^4 \ln(|\phi|^2/|\langle\phi\rangle|^2)$  is also obtained in the effective potential of the four-dimensional analogue of the Abelian Higgs model, the Coleman-Weinberg model. This anomalous term again originates from the gauge field and gives rise to a nontrivial vacuum expectation value, the first instance of the Coleman-Weinberg mechanism [225]. Further, similar nonanalyticities in field space to those encountered in this subsection will arise later in Sec. 2.3 and Sec. 3.4 in the context of Luttinger semimetals, again as a result of additional gapless modes, namely massless fermions. Lastly, note that the fixed-point collision in this subsection is qualitatively different from the fixed-point collision of

the Wilson-Fisher and Gaussian fixed points as discussed in Sec. 1.1.3. In the language of bifurcation theory, the former corresponds to a saddle-node bifurcation, meaning two fixed points annihilate after colliding, while the latter represents a transcritical bifurcation, meaning two fixed points do not annihilate but exchange stability in the collision direction [217, 226]. A visual comparison is provided in Fig. 1.6. As illustrated in Chap. 3, saddle-node bifurcations can be used to generate exotic critical behavior. Further, they share some of their phenomenology with quasiuniversal behavior from a marginally irrelevant operator, as discussed in Chap. 2.

## 1.4. Deconfined phase transitions

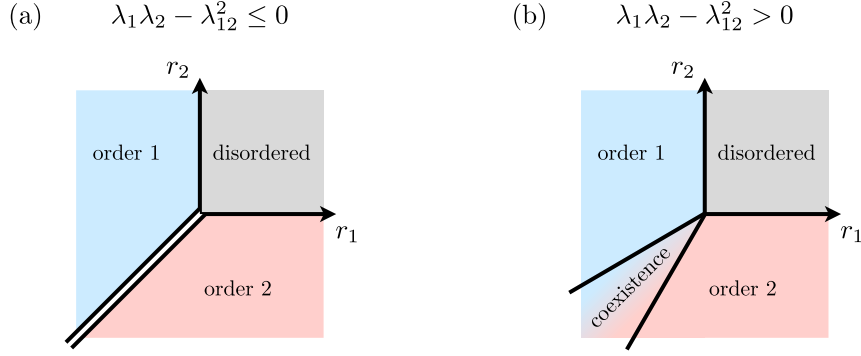
A third route beyond the Landau-Ginzburg-Wilson paradigm is constituted by a fractionalization mechanism that leads to generic continuous phase transitions between two phases that break distinct symmetries. We begin with a brief review of why such transitions lie beyond the Landau-Ginzburg-Wilson paradigm. We then introduce a fractionalization mechanism that enables generic continuous order-to-order phase transitions, and conclude by discussing its caveats.

### 1.4.1. Fine-tuned continuous order-to-order phase transitions

It is often said that continuous order-to-order phase transitions are forbidden in Landau-Ginzburg-Wilson theory. This is not (entirely) true, which we would like to demonstrate using the Landau potential [46, 227],

$$f_L = \frac{1}{2} (r_1 \phi_1^2 + r_2 \phi_2^2) + \frac{1}{4!} (\lambda_1 \phi_1^4 + 2\lambda_{12} \phi_1^2 \phi_2^2 + \lambda_2 \phi_2^4), \quad (1.18)$$

of two Ising scalars  $\phi_1$  and  $\phi_2$ , where  $r_i$  denote the corresponding tuning parameters and  $\lambda_1, \lambda_2 > 0$  ( $\lambda_{12} > -\sqrt{\lambda_1 \lambda_2}$ ) the symmetry-allowed coupling constants. The phase diagram is readily obtained by minimizing the above potential and schematically displayed in Fig. 1.7. For all values of the coupling constants, the phase diagram features a disordered phase and two ordered phases, each breaking exactly one of the symmetries,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ . The resulting transitions from the disordered phase into either of the ordered phases are continuous. For  $\lambda_1 \lambda_2 - \lambda_{12}^2 \leq 0$  the direct transition from one to the other ordered phase is discontinuous, unless we tune through the origin. For  $\lambda_1 \lambda_2 - \lambda_{12}^2 > 0$  an additional coexistence phase emerges and transitions into and out of this phase are continuous. In this case as well, the only way to obtain a continuous order-to-order transition is by tuning through the origin. We conclude that continuous order-to-order phase transitions can indeed occur within the above Landau theory, however, they always require fine tuning [227]. More generally, this statement holds for all other theories within the Landau-Ginzburg-Wilson paradigm, whose critical behavior is governed by Wilson-Fisher fixed points [56]. Whether



**Figure 1.7.:** Schematic phase diagram of the Landau theory for two Ising scalars as a function of the corresponding tuning parameters  $r_1$  and  $r_2$ . Continuous (discontinuous) transitions are indicated by single (double) black lines. (a) In case the couplings fulfill  $\lambda_1\lambda_2 - \lambda_{12}^2 \leq 0$ , we observe a disordered phase (gray) and two ordered phases (red and blue). (b) For  $\lambda_1\lambda_2 - \lambda_{12}^2 > 0$  an additional coexistence phase (color gradient) emerges. In both cases, a continuous order-to-order transition arises only when tuning through the origin. Figure adapted from Ref. [227].

a coexistence phase is present depends on the renormalization group flow topology of the specific model.

#### 1.4.2. Fractionalization mechanism

Building on the previous subsection, it is natural to ask whether continuous order-to-order phase transitions can also arise *generically*, i.e., without the need for fine tuning. To address this question, we consider the prominent example of  $SU(2)$  quantum spins on the square lattice, subject to short-range antiferromagnetic interactions, first in a Schwinger boson formulation and subsequently in a dual formulation.

##### Schwinger boson formulation

The antiferromagnetic  $SU(2)$  quantum spin system exhibits at least two phases as a function of spin exchange interactions: the antiferromagnetic Néel phase, which breaks spin-rotation, time-reversal, and translation symmetries while preserving all point-group symmetries and a combination of time reversal and translation, and the valence-bond-solid phase, which breaks some of the translation and point-group symmetries while leaving spin-rotation and time-reversal symmetries intact [161, 228, 229]. We emphasize that the symmetry group associated with the antiferromagnetic phase is not a subgroup of the symmetry group of the valence-bond-solid phase, and vice versa. Surprisingly, a direct continuous phase transition from antiferromagnetic to valence-bond-solid order has been proposed, captured by the noncompact  $CP^{N-1}$  model [228–230],

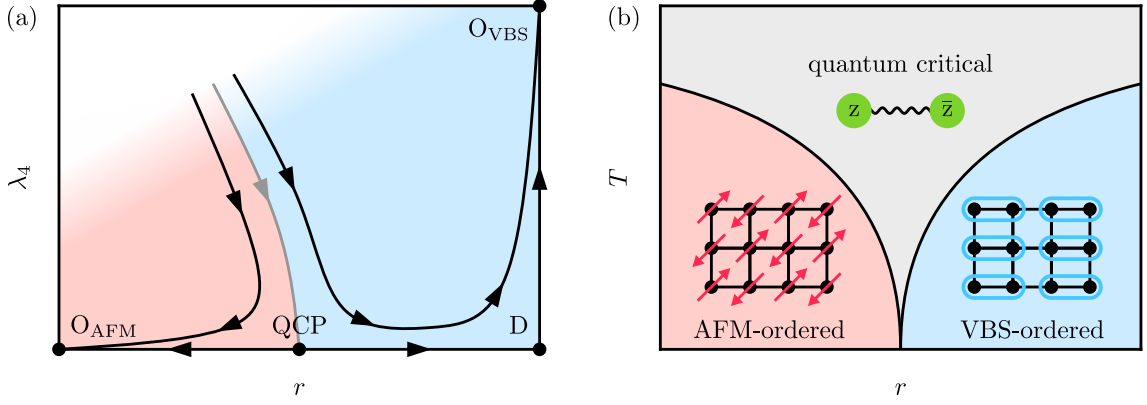
$$S_{CP^{N-1}} = \int d^3x \left[ |D_\mu z_i|^2 + \frac{1}{4} f_{\mu\nu}^2 \right], \quad (1.19)$$

where  $z_i$  denotes one of the  $N = 2$  fractionalized bosonic spinon fields subject to the constraint  $|z_i|^2 = 1$  and  $CP^{N-1} = S^{2N-1}/U(1)$  represents the complex projective space,

where  $S^{2N-1}$  is the unit  $(2N - 1)$ -sphere. The spinons are minimally coupled to an emergent  $U(1)$  gauge field  $a_\mu$ , incorporated in the field-strength tensor  $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ , via the covariant derivative  $D_\mu = \partial_\mu - ie a_\mu$  and the charge  $e$ . With monopole contributions argued to be irrelevant at the putative quantum critical fixed point, the gauge field  $a_\mu$  is taken to be noncompact [228, 229, 231–233]. The above theory follows from the Schwinger boson representation of a generalized antiferromagnetic  $SU(N)$  quantum spin system [234–236] and is, as such, formulated entirely in terms of new “deconfined” degrees of freedom, which confine in either phase [228, 229]. We emphasize that both phases are characterized by local order parameters. In the field theory, they correspond to the Néel vector  $z^\dagger \vec{T} z$  (monopole operator  $\mathcal{M}_a$ ) in case of the antiferromagnetic (valence-bond-solid) phase, where  $\vec{T}$  denotes the “vector” of generalized Gell-Mann matrices [237].

The existence of four equivalent columnar valence-bond-solid states may suggest that the antiferromagnetic-to-valence-bond-solid phase transition is described by a simple  $\mathbb{Z}_4$  clock model [161]. However, such a description would overlook the fact that vortices in the valence-bond-solid phase are point defects arising from the formation of domain walls and, as such, each carry one unpaired spin. Their proliferation therefore does not yield a noninteracting disordered phase, as in the three-dimensional XY model, but, likely, an antiferromagnetic ordered phase. In other words, the topological defects in the valence-bond-solid phase carry the quantum numbers of the antiferromagnetic Néel phase [161, 229, 238]. The converse also applies: skyrmion defects in the antiferromagnetic phase transform in the same way as the valence-bond-solid order parameter  $\mathcal{M}_a$ , thus can be viewed as precursors of the valence-bond-solid phase [161, 229, 238]. In sum, both phases appear closely intertwined, with each phase resulting from the proliferation of topological defects of the other, hinting at a direct continuous phase transition.

After presenting compelling intuitive evidence for the existence of a continuous order-to-order phase transition in antiferromagnetic  $SU(N)$  quantum spin systems, we now turn to a quantitative discussion. Softening the hard constraint  $|z_i|^2 = 1$  by introducing the potential  $V(z) = r|z_i|^2 + \lambda(|z_i|^2)^2$  turns the noncompact  $CP^{N-1}$  model into the familiar  $N$ -component Abelian Higgs model [241], Eq. (1.15). As discussed in Sec. 1.3.2 in detail, the Abelian Higgs model features a charged quantum critical fixed point for sufficiently large component numbers  $N \geq N_c$ , with  $N_c(D = 3) = 12.2 \pm 3.9$  being the state-of-the-art estimate from a  $4 - \epsilon$  expansion at the four-loop order [215]. Therefore, the intuitive arguments given above seem to apply for sufficiently large component numbers  $N$ , which is already enough to prove the concept of a continuous order-to-order phase transition arising from fractionalization. The behavior at smaller component numbers  $N < N_c$  is a separate question and we refer the reader to the subsection below for a discussion of various potential scenarios. The renormalization group flow associated with  $SU(N \geq N_c)$  quantum spin systems is visualized in Fig. 1.8(a). The flow in the subspace of vanishing 4-monopole fugacity,  $\lambda_4 = 0$ , follows directly from the  $N$ -component Abelian Higgs model, or equivalently its hard-constraint analogue, the noncompact  $CP^{N-1}$  model. However, to obtain the flow towards the infrared stable valence-bond-solid fixed point requires the



**Figure 1.8.:** (a) Schematic renormalization group flow diagram of  $SU(N)$  square lattice antiferromagnets, shown as a function of the tuning parameter  $r$  and the 4-monopole fugacity  $\lambda_4$ . Arrows indicate the flow towards the infrared. The infrared stable fixed point  $O_{\text{AFM}}$  ( $O_{\text{VBS}}$ ) gives rise to the antiferromagnetic Néel (valence-bond-solid) phase depicted in red (blue). The quantum critical fixed point QCP exists for sufficiently large  $N \geq N_c$  and renders the order-to-order transition continuous. The monopole fugacity  $\lambda_4$  is dangerously irrelevant at the QCP, thus destabilizes the  $U(1)$  spin liquid fixed point D. (b) Resulting schematic finite-temperature ( $T$ ) phase diagram depicting the continuous quantum phase transition from antiferromagnetic to valence-bond-solid order. While the two lower insets visualize the corresponding orders, the inset in the quantum critical regime displays a spinon interacting with its antiparticle via the exchange of a photon. Subfigures inspired by Ref. [230] and Refs. [239, 240], respectively.

inclusion of monopole events [229, 234, 235]. With a finite monopole fugacity  $\lambda_4 > 0$  in the ultraviolet [161, 230], a continuous phase transition from antiferromagnetic to valence-bond-solid order arises, as depicted in Fig. 1.8(b).

### A dual perspective

Apart from a conjectured strong self-duality, the noncompact  $CP^1$  model is also proposed to be dual, again in the strong sense, to a seemingly unrelated theory: three-dimensional quantum electrodynamics with supplemental Gross-Neveu interaction [242]. Within a partially bosonized formulation, the Euclidean action reads [242–244]

$$S_{\text{QED}_3\text{-GN}} = \int d^3x \left[ \bar{\psi}_i (\gamma_\mu D_\mu + g\phi) \psi_i + \frac{1}{4} f_{\mu\nu}^2 + \frac{1}{2} \phi (r - \partial_\mu^2) \phi + \lambda \phi^4 \right], \quad (1.20)$$

where  $\psi_i$  and  $\bar{\psi}_i$  denote  $N = 2$  flavors of two-component Dirac spinors, and the  $2 \times 2$  matrices  $\gamma_\mu$  satisfy the Clifford algebra  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \mathbb{1}$ . As before,  $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$  represents the field-strength tensor associated with the  $U(1)$  gauge field  $a_\mu$ , and  $D_\mu = \partial_\mu - ie a_\mu$  is the covariant derivative involving the charge  $e$ . Mediated by the Yukawa coupling  $g$ , the fermionic excitations are coupled to a real scalar field  $\phi$ , which possesses a mass  $r$  and is subject to a quartic self-interaction  $\lambda$ . The resulting chiral spin liquid (Dirac spin liquid) phase is signaled by broken (intact) time-reversal symmetry,  $\langle \phi \rangle \neq 0$  ( $\langle \phi \rangle = 0$ ) [245].

We point out this potential duality here for several reasons. First, if existent, it directly implies an emergent  $SO(5)$  symmetry at the quantum critical point [242], as observed in

numerical calculations [246]. Second, if this strong duality exists, it allows us to view the antiferromagnetic-to-valence-bond-solid phase transition from an entirely different perspective. This is a nontrivial statement: although three-dimensional quantum electrodynamics with supplemental Gross-Neveu interaction represents a strongly coupled problem, it is amenable to various approaches, including the  $4-\epsilon$  expansion [243–245] and the large- $N$  expansion [247–251]. While a quantum critical fixed point has been identified, it is not yet clear whether this fixed point undergoes a collision and annihilation with another fixed point, as suggested by the Abelian Higgs model, assuming the strong duality holds up to mid-infrared scales set by the finite correlation length [242, 244], cf. Sec. 1.4.3. Third, it is worth noting that many of the models encountered in the three research chapters of this dissertation are of similar structure to the theory defined above, i.e., they consist of fermions near band-touching points coupled to bosonic order-parameter fields and, in many cases, also to gauge fields. In particular, we return to three-dimensional quantum electrodynamics with supplemental four-fermion interactions in Chap. 3, where we propose an alternative mechanism for generic continuous order-to-order phase transitions.

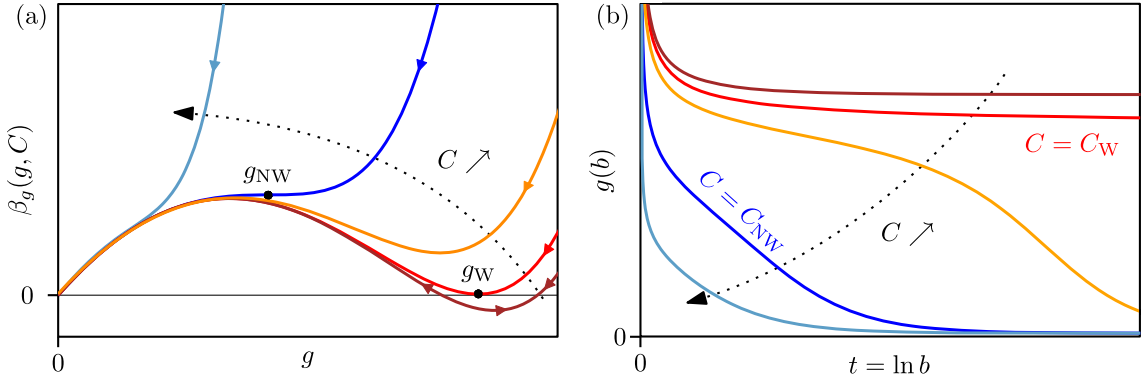
We would like to conclude this subsection with a few remarks. First, note that antiferromagnetic  $SU(2)$  quantum spin systems can also be studied within the Abrikosov fermion representation, which leads to three-dimensional quantum chromodynamics with two flavors and two colors as the effective description [242]. Second, beyond the antiferromagnetic-to-valence-bond-solid phase transition discussed above, other potential realizations of deconfined quantum critical points in two spatial dimensions have been proposed, including phase transitions between a quantum spin Hall insulator and an  $s$ -wave superconductor [252–255], or between various (semi)metals and Mott insulators [256–258]. Further, deconfined quantum critical points have also been proposed in other dimensions. The reader is referred to Refs. [259–263] and to Ref. [264] for discussions of deconfined criticality in one and three spatial dimensions, respectively.

### 1.4.3. Weakly first-order phase transitions

While the fractionalization mechanism discussed in the previous subsection certainly applies to  $SU(N)$  quantum spin systems with  $N \gg 2$ , the  $SU(2)$  case remains far less conclusive. Recent experimental [265, 266] and theoretical studies [153, 215, 233, 267–271] indicate a discontinuous order-to-order phase transition with an exceptionally small jump in the order parameter, a large yet finite correlation length, and an emergent approximate  $SO(5)$  symmetry. The purpose of this subsection is to outline two mechanisms that yield such generic weakly first-order phase transitions.

#### Quasiuniversality from fixed-point annihilation

The first mechanism was already encountered implicitly in Sec. 1.3.2, and is discussed here in more detail. Let us recall that the  $N$ -component Abelian Higgs model, the soft-



**Figure 1.9.:** (a) Family of curves showing the beta function  $\beta_g$  as a function of the coupling  $g$  for different values of an external parameter  $C$ , such as the inverse of the number of order-parameter components or fermion flavors,  $C = 1/N$ , or the Wess-Zumino-Witten level,  $C = k$ . (b) Corresponding solutions, i.e.,  $g$  as a function of the renormalization group time  $t = \ln b$ , for different values of the external parameter  $C$  and for sufficiently large couplings at the ultraviolet scale  $b_{UV} = 1$ . For  $C \gtrsim C_W$ , the beta function possesses two complex interacting fixed points with small imaginary parts originating from a fixed-point annihilation at  $C = C_W$ . This leads to a logarithmically slow walking coupling and quasiuniversal behavior. For  $C \approx C_{NW} > C_W$ , the system is close to a saddle point originating in the collision of two local extrema. Again, the renormalization group flow becomes logarithmically slow, yet faster than in the walking regime, a phenomenon termed Nordic-walking behavior. Figure adapted from Ref. [271].

constraint analogue of the noncompact  $CP^{N-1}$  model, features an annihilation of the critical and multicritical charged fixed points when lowering the number of order-parameter components  $N$  below the critical number  $N_c$ , as visualized in Fig. 1.5(c) and Fig. 1.6(a). Slightly below the critical component number,  $N \lesssim N_c$ , both complex fixed points possess only small imaginary parts, i.e., they are still reasonably close to the physical space of real couplings and give rise to a region in theory space characterized by a logarithmically slow “walking” renormalization group flow [272, 273], as depicted in Fig. 1.9. This results in an exponentially large, though finite, correlation length and power-law behavior with slowly varying exponents just below the critical component number  $N_c$ , a phenomenon termed quasiuniversal, or pseudocritical, behavior. [215, 233, 242, 274, 275]. By lowering the number of order-parameter components  $N$  further, one encounters a hierarchy of scales, say  $\xi(\lfloor N_c \rfloor) \gg \xi(\lfloor N_c \rfloor - 1) \gg \xi(\lfloor N_c \rfloor - 2) \gg \dots$ , since the dimensionless correlation length  $\xi$  depends exponentially on the deviation from the critical component number  $N_c$ ,

$$\xi \simeq \exp\left(\frac{c}{\sqrt{N_c - N}}\right), \quad (1.21)$$

where  $c > 0$  denotes a positive constant [215, 233, 272, 276]. Besides the Abelian Higgs model, Miransky scaling is known from a broad range of systems, such as Potts models [277–280], Wess-Zumino-Witten models [271, 274, 275], Luttinger semimetals [281–283], quantum electrodynamics in three space-time dimensions [217, 284–286], quantum impurity models [287–290], and quantum chromodynamics in four space-time dimensions [276, 291–294]. Except for Potts models, we will return to all realizations, in the order presented above, throughout this dissertation.

Let us come back to  $SU(2)$  quantum spin systems. The state-of-the-art estimate for the

critical component number  $N_c(D = 3) = 12.2 \pm 3.9$  being well above the physical value  $N = 2$  suggests that the antiferromagnetic-to-valence-bond-solid phase transition may not be truly continuous, but rather weakly first order [215]. This resonates, for instance, with the absence of an  $SO(5)$ -symmetric conformal field theory (with compatible exponents) from the conformal bootstrap program [97, 267]. Nevertheless, we emphasize that quasi-universality as discussed here is still a nontrivial behavior beyond the Landau-Ginzburg-Wilson paradigm, since the latter does not encompass exotic scenarios such as fixed-point annihilations. As pointed out in the previous subsection, a key open question is whether a “dual” fixed-point annihilation occurs in the nonperturbative regime of quantum electrodynamics with supplemental Gross-Neveu interaction [242, 244]. Before turning to the next paragraph, which offers an alternative view on  $SU(2)$  quantum spin systems, we would like to point out that there exists a different mechanism for quasiuniversality, independent of fixed-point annihilation and rooted in the presence of a marginally irrelevant coupling at a real quantum critical fixed point, see Chap. 2.

### Nordic-walking mechanism

Instead of working with slave-particles formulations and their dualities, one may attempt to approach the problem by employing the functional integral description of quantum spin systems [105, 295]. In the present case of  $SU(2)$  spins on the square lattice subject to short-range antiferromagnetic interactions, this yields an  $O(5)$  nonlinear sigma model in three space-time dimensions supplemented with a Wess-Zumino-Witten term at level  $k = 1$  [271, 275, 296, 297],

$$S_{\text{NLSM-WZW}} = \frac{1}{2g} \int d^3x (\partial_\mu \phi)^2 + \frac{2\pi i k \epsilon^{a_1 \dots a_5}}{S_4} \int_0^1 du \int d^3x \phi_{a_1} \partial_{x_1} \phi_{a_2} \dots \partial_u \phi_{a_5}, \quad (1.22)$$

where  $\phi$  denotes a suitably extended constrained vector field,  $\phi^2 = 1$ , with three (two) of its five real components corresponding to the antiferromagnetic (valence-bond-solid) order parameter. The Wess-Zumino-Witten term captures the intertwinement of the antiferromagnetic and valence-bond-solid orders by measuring the ratio of the surface area traced out by a specific field configuration to the surface area  $S_4$  of the unit 4-sphere, but, due to its topological nature, prohibits standard approaches such as the  $\epsilon$  expansion. This can be circumvented by employing a nonperturbative renormalization group calculation, which indeed shows a collision and subsequent annihilation of the critical fixed point with an unstable fixed point upon increasing the level  $k$  [271, 274, 275]. However, the critical value  $k_W \approx 0.37$  seems to be too small to explain the quasiuniversal behavior through fixed-point annihilation in the most interesting case  $k = 1$  [271]. Instead, the calculation suggests that the logarithmically slow renormalization group flow arises from the collision of local extrema, resulting in a saddle point of the beta function  $\beta_g$ , as displayed in Fig. 1.9(a). In the vicinity of this saddle node, the beta function is approximately flat over an extended region, which also results in a logarithmically slow renormalization group flow, here setting in around  $k_{\text{NW}} \approx 0.79$  [271]. As visualized in

Fig. 1.9(b), a coupling in the vicinity of a saddle point flows more slowly than a running coupling, yet faster than the walking behavior associated with fixed-point annihilations as discussed above, hence the term Nordic-walking behavior. In the physical case of level  $k = 1 \gtrsim k_{\text{NW}}$ , the Nordic-walking mechanism offers an alternative explanation for the slowly varying exponents and the emergent approximate  $\text{SO}(5)$  symmetry observed in experiments [265, 266] and numerical studies [153, 233, 269]. Despite its slightly faster flow, the Nordic-walking mechanism has the advantage that it does not rely exclusively on fixed-point annihilations but occurs whenever the beta function exhibits a flat region. Apart from its condensed-matter applications, the Nordic-walking mechanism, similarly to the fixed-point-annihilation mechanism, offers a natural explanation to hierarchy problems in particle physics and cosmology, whether it is realized in nature remains a question for future research [271, 298].

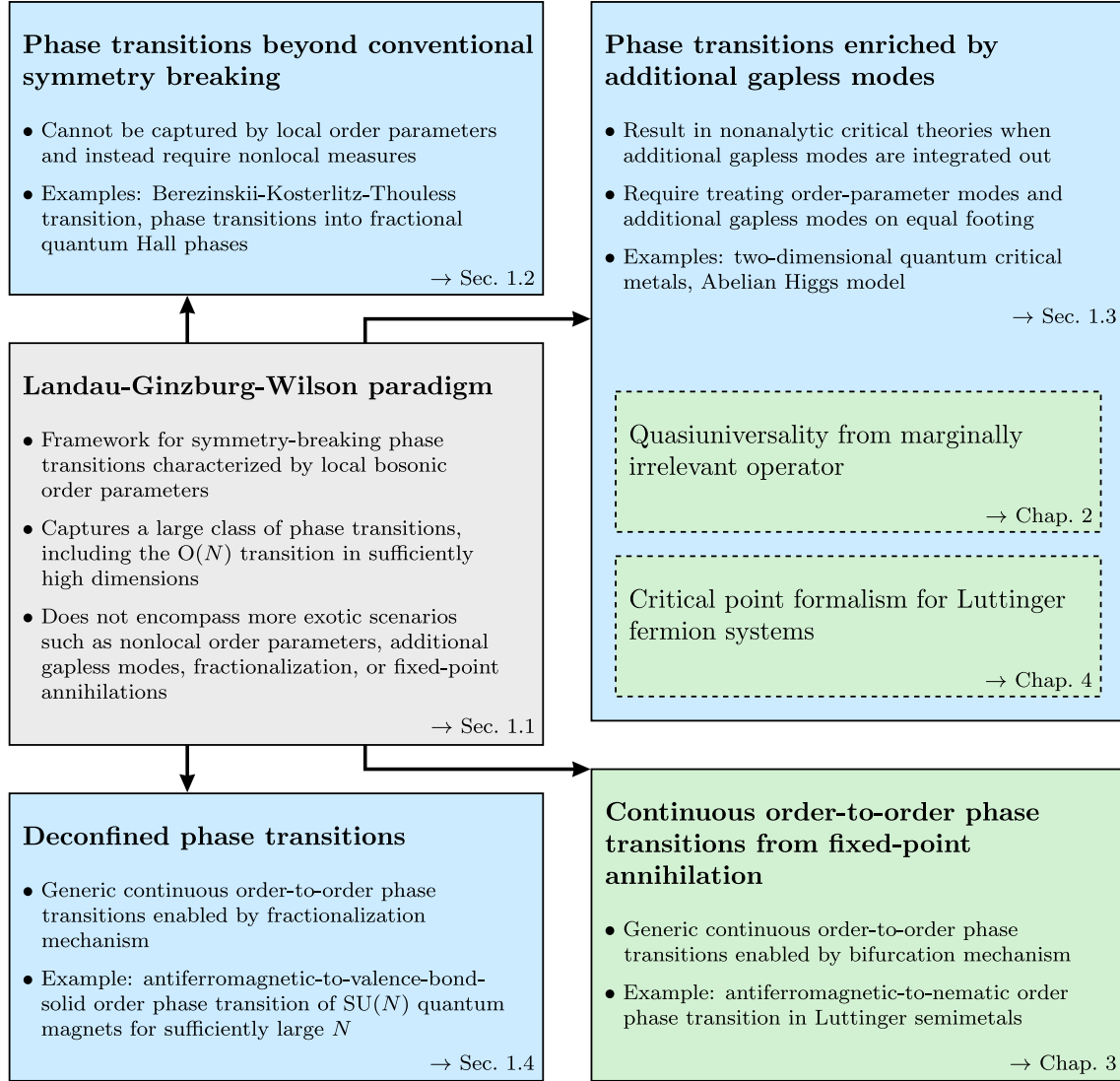
## 1.5. Dissertation overview

Having reviewed the Landau-Ginzburg-Wilson paradigm in Sec. 1.1 and explored various routes beyond it in the subsequent Secs. 1.2–1.4, we now turn our attention to the research part of this dissertation, which is structured as follows.

In Chap. 2, the first of three research chapters, we propose a novel mechanism for quasiuniversality and exemplify it in the context of Luttinger semimetals, which are charged fermionic systems capable of undergoing topological phase transitions and thus closely connected to the concepts encountered in Secs. 1.2–1.3. Additionally, the proposed mechanism shares some of its phenomenology with deconfined pseudocriticality, cf. Sec. 1.4.3, however, in this case it originates from an exotic quantum critical fixed point. We discuss the experimental implications of this fixed point, along with recent experimental evidence for quasiuniversal behavior in the rare-earth pyrochlore iridates.

In Chap. 3, we introduce a novel mechanism for generic continuous order-to-order phase transitions and demonstrate its relevance across a wide range of systems, including Luttinger semimetals, kagome quantum magnets, quantum impurity models, and quantum chromodynamics. These models are again closely related to the models and concepts encountered in Secs. 1.2–1.3. The proposed mechanism offers an alternative to the fractionalization mechanism for deconfined criticality outlined in Sec. 1.4.2, as both mechanisms share key aspects of their phenomenology. In addition to relating this mechanism to established theoretical concepts, we discuss its characteristic experimental signatures.

In Chap. 4, we introduce a novel computational framework by extending the relativistic large- $N$  critical point formalism, designed to compute the critical exponents of relativistic critical fixed points perturbatively in  $1/N$ , to nonrelativistic field theories with Fermi nodes. Specifically, we investigate the large- $N$  behavior of fermions close to  $N$  quadratic band touching points, coupled to a superconducting order parameter, cf. Sec. 1.3. In the course of this analysis, we also explore relativistic and random Luttinger fermion systems



**Figure 1.10.:** Flow chart illustrating various routes beyond the Landau-Ginzburg-Wilson paradigm (gray box). Three routes (blue boxes) are illustrated in this introductory chapter, while a fourth (single green box with solid edges) is proposed and exemplified in one of the three research chapters. The two remaining research chapters propose a new mechanism and a new computational technique (embedded green boxes with dashed edges), respectively, both of which are applicable to critical theories enriched by additional gapless modes. This flow chart is not intended to be exhaustive and is simplified, for instance, as some areas may overlap, but aims to provide a rough overview of the field and the structure of this thesis.

by means of critical point techniques. In the spirit of Witten’s quote at the beginning of this chapter, we maintain the focus on the underlying concepts rather than on mathematical details.

The dissertation concludes with general remarks, presented in Chap. 5. A visual overview of the thesis and the key concepts encountered or developed throughout is given in Fig. 1.10.



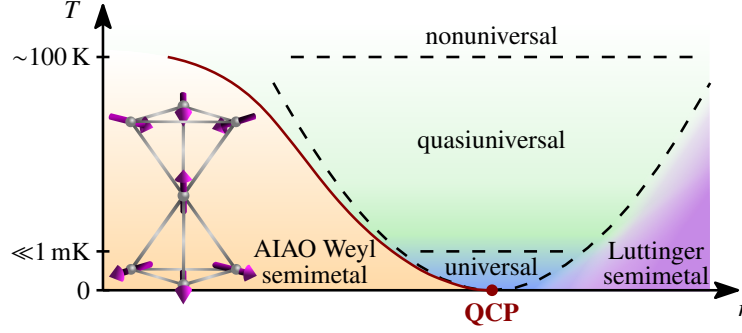
## 2. Quasiuniversality from marginally irrelevant operator

*“It’s an experience like no other experience I can describe, the best thing that can happen to a scientist, realizing that something that’s happened in his or her mind exactly corresponds to something that happens in nature. It’s startling every time it occurs. One is surprised that a construct of one’s own mind can actually be realized in the honest-to-goodness world out there. A great shock, and a great, great joy.” [299]*

— Leo P. Kadanoff

Quasiuniversal behavior is characterized by power laws with exponents that vary slowly over several orders of magnitude in energy or length, and has been extensively discussed in the context of deconfined criticality. In this chapter, we propose a novel mechanism for quasiuniversality, based on the presence of a marginally irrelevant operator at the quantum critical point. We exemplify this mechanism by identifying exotic quasiuniversal behavior near the all-in-all-out (AIAO) Weyl quantum critical point in three-dimensional Luttinger semimetals, such as the pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ , with  $R$  denoting a rare-earth element. In this limit, the pertinent critical exponents can be computed exactly within a renormalization group analysis. Experimental implications for the pyrochlore iridates are outlined and compared with recent experimental findings, which appear consistent with out theoretical predictions.

This chapter is organized as follows. First, we motivate this project, see Sec. 2.1, and then introduce a Luttinger semimetal field theory suitable for illustrating the proposed mechanism, see Sec. 2.2. Second, we study its universal properties through a mean-field analysis, see Sec. 2.3, and a renormalization group analysis, see Sec. 2.4. Third, we turn to the quasiuniversal regime, see Sec. 2.5, and comment on its experimental signatures as well as recent experimental results for the rare-earth pyrochlore iridates, see Sec. 2.6. We conclude in Sec. 2.7.

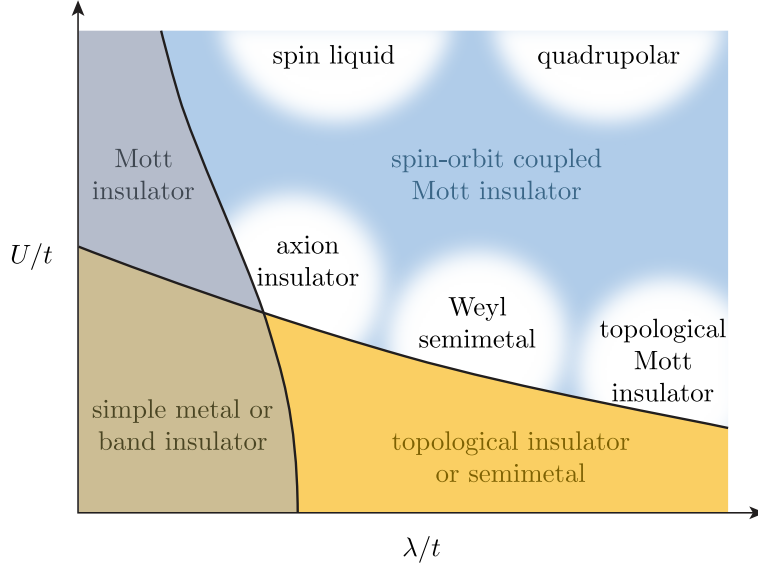


**Figure 2.1.:** Schematic finite-temperature phase diagram of rare-earth pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$  near the quantum critical point (QCP) between the Weyl semimetal with all-in-all-out (AIAO) order, sketched in the inset, and the symmetric Luttinger semimetal.  $r$  indicates a nonthermal tuning parameter, such as chemical doping or hydrostatic pressure. As a result of a marginally irrelevant coupling at the quantum critical point, there is a large quasiuniversal regime, characterized by power laws with slowly varying exponents, between the nonuniversal high-temperature regime and the genuinely universal ultra-low-temperature regime. Figure reprinted from Ref. [301].

## 2.1. Motivation

### 2.1.1. Conceptual motivation

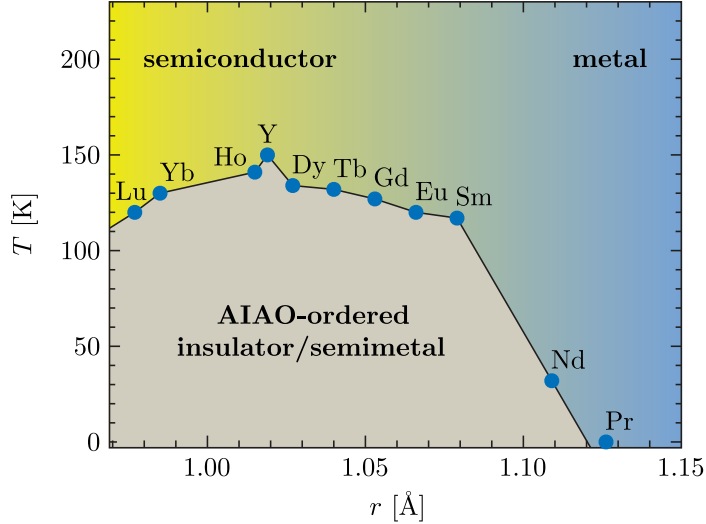
Quasiuniversality refers to a situation in which observables display apparent critical behavior over several orders of magnitude of energy or length, where, however, closer inspections of the observed power laws reveal slow drifts in the corresponding exponents [242, 274, 275]. As explained in Sec. 1.4.3, such a situation has recently been intensely discussed in the context of deconfined quantum criticality between antiferromagnetic and valence-bond-solid orders in two-dimensional quantum magnets [228, 229]. In this case, the quasiuniversal behavior observed in numerical simulations [233] is believed to arise from a collision and subsequent annihilation of the corresponding critical fixed point with another, bicritical, fixed point, leaving behind a slow renormalization group flow [215, 242, 274, 275]. In this scenario, the ultra-low-temperature behavior is ultimately weakly first order, with, however, a large finite-temperature regime characterized by quasiuniversality. An alternative interpretation of the numerical data is the presence of a second divergent length scale at criticality [300]. In this competing scenario, the low-temperature behavior is genuinely universal, but requires an adapted scaling ansatz accommodating the presence of the additional length scale. The goal of this chapter is to introduce and illustrate a different mechanism for quasiuniversality, giving rise to the same phenomenology in the intermediate-to-low-energy regime. In contrast to quasiuniversality arising from fixed-point annihilation, the ultra-low-temperature regime here will be governed by a genuine quantum critical fixed point with a marginally irrelevant operator. This novel mechanism thus offers a scenario where the observation of power-law behavior with drifting exponents is not only compatible with a genuine quantum critical fixed point, but is in fact the very result of an exotic quantum critical fixed point. The resulting finite-temperature phase diagram is sketched in Fig. 2.1, with a realization in the rare-earth pyrochlore iridates in mind, cf. Sec. 2.1.2.



**Figure 2.2.:** Schematic zero-temperature phase diagram for materials subject to the Hubbard repulsion  $U/t$  and the spin-orbit coupling  $\lambda/t$ , expressed in units of the hopping amplitude  $t$ . While not necessarily perfectly accurate, this introductory graphic is meant to highlight that electronic interactions and spin-orbit coupling, in this context, act not in simple competition, but as cooperative interplay, yielding various exotic phases of matter. For the pyrochlore iridates, this includes axion insulator [304–306], Weyl semimetal [304, 307], topological Mott insulator [203, 281, 308, 309], and chiral spin liquid [310] phases, to name a few. Figure adapted from Ref. [303].

### 2.1.2. Material motivation

We illustrate our mechanism by identifying quasiuniversal behavior in a three-dimensional model relevant to a class of pyrochlore iridates with chemical composition  $R_2\text{Ir}_2\text{O}_7$ , where  $R$  denotes a rare-earth element, e.g.,  $R = \text{Pr}, \text{Nd}$ . The rare-earth pyrochlore iridates, and more generally the whole pyrochlore-oxide family  $A_2B_2\text{O}_7$ , are materials known for realizing exotic states of matter that arise from the interplay of strong electronic interactions, spin-orbit coupling, and geometric frustration [302, 303], see Fig. 2.2. In their metallic phases, the rare-earth pyrochlore iridates exhibit an electronic excitation spectrum characterized by quadratic band touching at the Fermi level [311, 312], and as such fall into the larger class of Luttinger semimetals [313, 314]. However, in comparison with the prominent members of this class,  $\text{HgTe}$  [315] and  $\alpha\text{-Sn}$  [316], the pyrochlore iridates feature a substantially increased effective quasiparticle mass, implying an enhanced role of electronic interactions [317, 318]. If strong enough, these interactions can drive symmetry-breaking transitions, across which the electronic spectrum becomes partially or fully gapped out. Many pyrochlore iridates, such as  $\text{Nd}_2\text{Ir}_2\text{O}_7$  [312, 319], indeed display a finite-temperature transition, below which the iridium moments feature AIAO antiferromagnetic order, as depicted in Fig. 2.3. This state breaks time reversal but preserves crystal symmetries [303].  $\text{Pr}_2\text{Ir}_2\text{O}_7$ , by contrast, appears to remain disordered up to the lowest accessible temperatures [317, 322]. By varying the concentration  $x$  in  $(\text{Pr}_x\text{Nd}_{1-x})_2\text{Ir}_2\text{O}_7$  and/or by applying hydrostatic pressure, the Néel temperature associated with the onset of AIAO order can be tuned to zero, uncovering an underlying quantum phase transition [323]. Small AIAO order converts the quadratic band touching point into eight symmetry-related lin-



**Figure 2.3.:** Finite-temperature ( $T$ ) phase diagram for the pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$  as a function of the  $R^{3+}$  ionic radius  $r$ . Except for  $\text{Pr}_2\text{Ir}_2\text{O}_7$ , all larger pyrochlore iridates undergo a continuous phase transition from the high-temperature paramagnetic phase to the low-temperature AIAO-ordered phase. From the experimental side, the electronic structure of the AIAO-ordered phase appears to be insulating or semimetallic [320]. Figure adapted from Ref. [303], based on the thermodynamic and transport measurements of Ref. [321].

ear band crossing points [307, 324, 325]. Consequently, the quantum phase transition is expected to separate the symmetric Luttinger semimetal from a time-reversal-broken Weyl semimetal. The presence of gapless fermions at the transition indicates the possibility of unconventional critical behavior. In fact, previous theoretical work suggested a novel type of fermionic quantum critical point [326, 327]. Here, we elucidate the finite-temperature properties of this transition, relevant for the experiments on the pyrochlore iridates, cf. Sec. 2.6. We contribute to their understanding by revealing a large finite-temperature regime above the quantum critical point that is characterized by quasiuniversality, see Fig. 2.1. This unusual behavior arises from the presence of a marginally irrelevant coupling at the corresponding renormalization group fixed point. Importantly, we present an approach that allows us to identify properties of this fermionic quantum critical point exactly, including its nontrivial critical exponents. Our results reveal the AIAO Weyl quantum critical point in pyrochlore iridates as a unique instance of an interacting continuous quantum phase transition that is both experimentally and theoretically accessible. As a side product, we demonstrate the emergence of strong cubic anisotropy at criticality, thereby resolving an apparent contradiction in the literature [326, 327].

## 2.2. Luttinger semimetals

At the quantum critical point, the system can be effectively described by a continuum Euclidean action  $S = \int d\tau d^3x \mathcal{L}$ , with Lagrangian  $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_a + \mathcal{L}_\phi$ . Here,  $\mathcal{L}_0$  describes

noninteracting electronic quasiparticles near the quadratic band touching [281, 328–332],

$$\mathcal{L}_0 = \sum_{i=1}^N \psi_i^\dagger \left[ \partial_\tau + \sum_{a=1}^5 (1 + s_a \delta) d_a (-i\nabla) \gamma_a \right] \psi_i, \quad (2.1)$$

and originates in the Luttinger Hamiltonian [333]. In the above,  $N$  corresponds to the number of band touching points at the Fermi level, with  $N = 1$  in the case of the pyrochlore iridates,  $\psi_i$  is a four-component Grassmann field,  $s_a := 1$  ( $s_a := -1$ ) for  $a = 1, 2, 3$  ( $a = 4, 5$ ),  $\delta$  parametrizes the cubic anisotropy with  $-1 \leq \delta \leq 1$ , the  $4 \times 4$  matrices  $\gamma_a$  fulfill the Euclidean Clifford algebra,  $\{\gamma_a, \gamma_b\} = 2\delta_{ab}\mathbb{1}$ , and  $d_a$  are proportional to  $\ell = 2$  real spherical harmonics, viz.  $d_1(\vec{p}) = \sqrt{3}p_y p_z$ ,  $d_2(\vec{p}) = \sqrt{3}p_x p_z$ ,  $d_3(\vec{p}) = \sqrt{3}p_x p_y$ ,  $d_4(\vec{p}) = \frac{\sqrt{3}}{2}(p_x^2 - p_y^2)$ , and  $d_5(\vec{p}) = \frac{1}{2}(2p_z^2 - p_x^2 - p_y^2)$ . Moreover, we account for the long-range Coulomb interaction  $\propto 1/|\vec{x}|$  via [281, 282, 334]

$$\mathcal{L}_a = \frac{1}{2}(\nabla a)^2 + iea \sum_{i=1}^N \psi_i^\dagger \psi_i, \quad (2.2)$$

where  $a$  denotes the scalar Coulomb field, and  $e$  the effective charge. Fluctuations corresponding to AIAO ordering on the pyrochlore lattice are parametrized by an Ising field  $\phi$ , and couple to the electronic quasiparticles as [326, 327, 335]

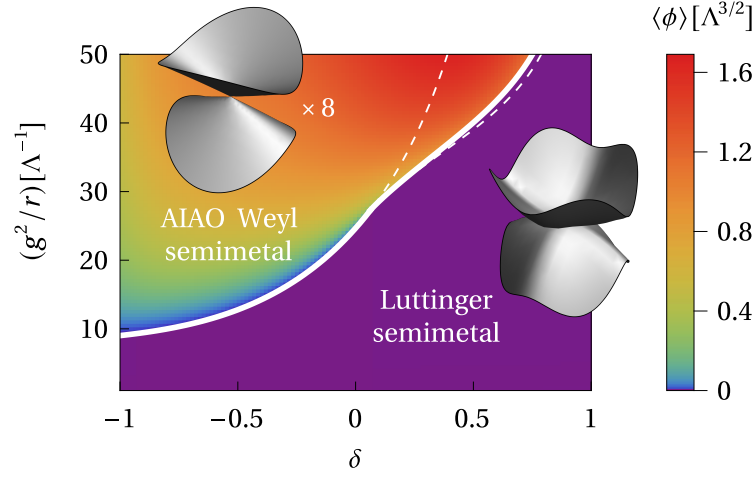
$$\mathcal{L}_\phi = \frac{1}{2}\phi(r - \nabla^2)\phi + g\phi \sum_{i=1}^N \psi_i^\dagger \gamma_{45} \psi_i. \quad (2.3)$$

Here,  $r$  denotes the tuning parameter for the quantum phase transition,  $g$  the coupling constant, and  $\gamma_{45} := i\gamma_4\gamma_5$ . A term  $(\partial_\tau \phi)^2$  can be included in Eq. (2.3) as well, but is power-counting irrelevant and as such does not change the critical behavior. The same is true for bosonic self-interactions, such as a  $\phi^4$  term. A finite expectation value  $\langle \phi \rangle \neq 0$  breaks time reversal and splits the quadratic band touching point into four pairs of Weyl nodes along the [111] and symmetry-related axes in the cubic basis.

It is worth noting that similar theories to the one defined above can also arise in high-energy physics. Like their nonrelativistic counterparts, relativistic Luttinger fermion theories exhibit many striking features, including the realization of asymptotic freedom already in the fermionic matter sector [336–338]. See Sec. 4.3 for a discussion of further exotic features of relativistic Luttinger fermions as well as their treatment within the large- $N$  critical point formalism.

## 2.3. Mean-field analysis

We start by discussing the model on the level of mean-field theory at zero temperature. Formally, mean-field theory corresponds to the limit of large number  $N$  of quadratic band touching points at the Fermi level, which effectively suppresses fluctuations of the bosonic fields. The mean-field energy is obtained by integrating out the fermions, leading to the



**Figure 2.4.:** Zero-temperature phase diagram of effective model as a function of cubic anisotropy  $\delta$  and interaction strength  $g^2/r$ , from mean-field theory. Coloring indicates magnitude of AIAO order parameter  $\langle\phi\rangle$ . With the onset of order, the quadratic band touching point of the Luttinger semimetal splits into four pairs of Weyl points in the AIAO Weyl semimetal, see insets. The transition is continuous (discontinuous) for  $\delta \leq \delta_0$  ( $\delta > \delta_0$ ), with  $\delta_0 \approx 0.0624$ . Dashed lines delimit the region in which metastable states exist. Figure reprinted from Ref. [301].

usual logarithm of the fermion determinant in the effective action for the order parameter. Performing the frequency integration leads to the mean-field energy

$$E_{\text{MF}}(\phi) = \frac{r}{2}\phi^2 + \int_0^\Lambda \frac{d^3p}{(2\pi)^3} \left[ \varepsilon_\phi^+(\vec{p}) + \varepsilon_\phi^-(\vec{p}) \right], \quad (2.4)$$

in accordance with Refs. [331, 339]. While the first summand penalizes the presence of a finite order parameter, the second one accounts for a lowering of the total energy by AIAO ordering, arising from the reduction of the low-energy density of states. The momentum integration is carried out up to an ultraviolet cutoff  $\Lambda$ , and is performed over the two filled fermionic bands

$$\varepsilon_\phi^\pm(\vec{p}) = - \left\{ (1-\delta)^2 p^4 + 4\delta \sum_{i=1}^3 d_i^2(\vec{p}) + (g\phi)^2 \pm 2(1+\delta)|g\phi| \left[ \sum_{i=1}^3 d_i^2(\vec{p}) \right]^{1/2} \right\}^{1/2}, \quad (2.5)$$

which are obtained by diagonalizing the mean-field Hamiltonian

$$H_{\text{MF}} = \sum_{a=1}^5 (1 + s_a \delta) d_a(\vec{p}) \gamma_a + g\phi \gamma_{45}. \quad (2.6)$$

Due to the nontrivial anisotropy dependence of the integrand Eq. (2.5), the integral Eq. (2.4) has to be performed numerically. Minimizing the resulting mean-field energy yields the phase diagram presented in Fig. 2.4.

The mean-field phase diagram displays two distinct phases, the paramagnetic Luttinger semimetal phase and the time-reversal-broken Weyl semimetal phase. The former is located at small coupling below a finite threshold  $(g^2/r)_c$  and hosts a quadratic band touching point with four-fold degeneracy at zero momentum. The Weyl semimetal phase, char-

acterized by a finite order parameter, is encountered above the phase boundary. It hosts four pairs of Weyl nodes along the  $[111]$  and symmetry-related axes in the electronic spectrum, and features AIAO order on the pyrochlore lattice. For anisotropies  $\delta \leq \delta_0 \approx 0.0624$ , we observe a continuous phase transition, while for  $\delta > \delta_0$ , the phase transition becomes discontinuous. The latter case gives rise to metastable states in the region around the transition, delimited by the dashed lines in Fig. 2.4. Importantly, ab-initio calculations and photo emission spectroscopy experiments in  $\text{Pr}_2\text{Ir}_2\text{O}_7$  [311] suggest  $\delta < 0$ , placing this material into the regime with continuous transition<sup>1</sup>. Note that at strong coupling, a second transition towards a Mott-insulating phase may be expected [304, 307, 325], which is not captured by the mean-field theory of our effective model. However, the presence or absence of this strong-coupling phase is irrelevant for the physics close to the quantum critical point between Luttinger and AIAO Weyl semimetals, which we will focus on in the following.

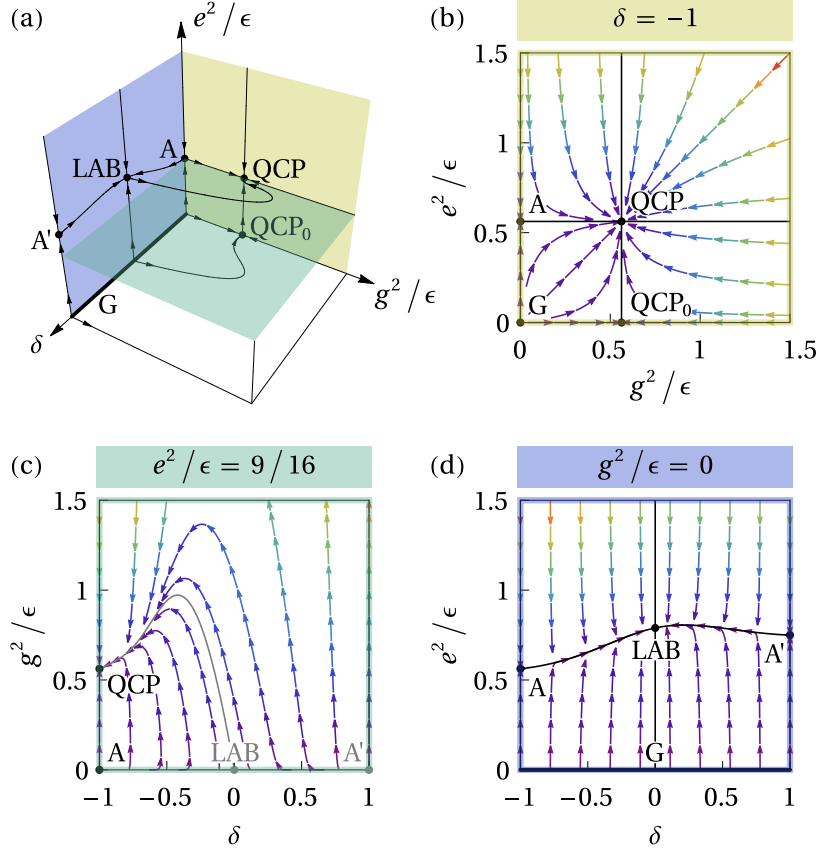
## 2.4. Renormalization group analysis

After discussing the mean-field behavior in the previous section, we now turn to the regime of strong order-parameter fluctuations by employing a renormalization group approach. To keep the presentation as nontechnical as possible, we first summarize the main results of this analysis in Sec. 2.4.1, and then provide further details for the interested reader. In particular, we present the derivation of the flow equations in Sec. 2.4.2, a discussion of the emergent particle-hole symmetry at the quantum critical point in Sec. 2.4.3, an overview of the fixed-point structure in Sec. 2.4.4, and an investigation of higher-loop corrections in Sec. 2.4.5.

### 2.4.1. Essentials

The partially bosonized Lagrangian  $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_a + \mathcal{L}_\phi$ , Eqs. (2.1)–(2.3), features a unique upper critical dimension, since both the Yukawa coupling  $g$  and the effective charge  $e$  become marginal in  $d = 4$  spatial dimensions. This allows a standard  $\epsilon = 4 - d$  expansion. We start by discussing the results at one-loop order, naively valid only for small  $\epsilon$ , but will then show that higher-loop corrections in fact vanish exactly at the AIAO Weyl quantum critical point. As will be derived in Sec. 2.4.2, integrating out modes with momenta  $q$  in the thin shell  $\Lambda/b < q < \Lambda$ , where  $\Lambda$  denotes the ultraviolet cutoff, and arbitrary frequency,

<sup>1</sup>The hopping parameters in the lattice model of Ref. [324] indicate  $\delta \approx -0.07$ , which is qualitatively consistent with the Tran-Blaha modified Becke-Johnson (TB-mBJ) calculations of Ref. [311]. We thank Igor Boettcher for helpful communication on this point.



**Figure 2.5.:** Fixed-point structure for  $N = 1$  on the critical hypersurface  $r = 0$  in (a) the parameter space spanned by  $\delta$ ,  $g^2$ , and  $e^2$ , (b) the  $g^2$ - $e^2$  plane for fixed  $\delta = -1$ , (c) the  $\delta$ - $g^2$  plane for fixed  $e^2 = 9\epsilon/16$ , and (d) the  $\delta$ - $e^2$  plane for fixed  $g^2 = 0$ . Black (gray) dots and lines indicate fixed points and separatrices, respectively, located within (projected onto) the respective planes. Arrows denote flow towards the infrared, with their coloring indicating the flow velocity. There is a unique quantum critical fixed point at finite coupling, labeled as QCP. It corresponds to the continuous transition between the symmetric Luttinger semimetal and the AIAO-ordered Weyl semimetal. The symmetric state is described by the Luttinger-Abrikosov-Beneslavskii fixed point at  $g^2 = 0$ , labeled as LAB. The dots labeled as A, A', QCP<sub>0</sub>, and G correspond to unstable interacting and Gaussian, respectively, fixed points, further discussed in Sec. 2.4.4. Figure reprinted from Ref. [301].

leads to the flow equations at criticality  $r = 0$

$$\frac{d\delta}{d \ln b} = -\frac{2}{15}(1 - \delta^2)^2 \left[ (g^2 + e^2)f_{1e} + (g^2 - e^2)f_{1t} \right], \quad (2.7)$$

$$\begin{aligned} \frac{dg^2}{d \ln b} = & (\epsilon - \eta_\phi)g^2 - \frac{2}{15}(1 - \delta^2) \left[ (1 + \delta)(g^2 + e^2)f_{1e} \right. \\ & \left. - (1 - \delta)(g^2 - e^2)(f_{1t} - 3f_{2e}) \right] g^2, \end{aligned} \quad (2.8)$$

$$\begin{aligned} \frac{de^2}{d \ln b} = & (\epsilon - \eta_a)e^2 - \frac{2}{15}(1 - \delta^2) \left[ (1 + \delta)(g^2 + e^2)f_{1e} \right. \\ & \left. - (1 - \delta)(g^2 - e^2)f_{1t} \right] e^2, \end{aligned} \quad (2.9)$$

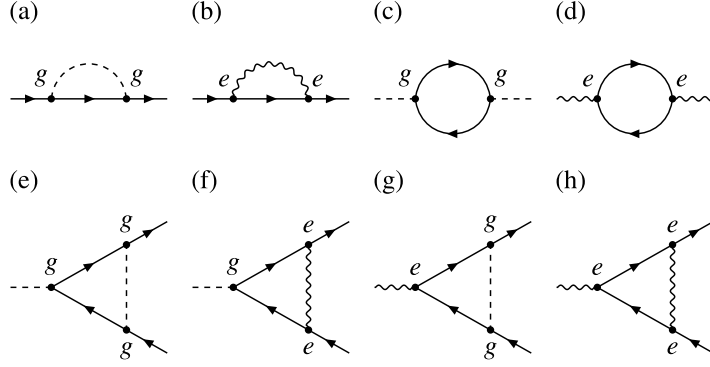
where  $\eta_\phi = Ng^2f_{g^2}$  and  $\eta_a = Ne^2f_{e^2}$ . Here,  $f_i \equiv f_i(\delta)$  with  $i \in \{1e, 1t, 2e, e^2\}$  ( $i \in \{g^2\}$ ) are bounded and continuous functions of the anisotropy parameter  $\delta$  with  $f_i > 0$  ( $f_{g^2} \geq -2/3$ ) for  $\delta \in [-1, 1]$  and  $f_i = 1$  ( $f_{g^2} = 0$ ) for  $\delta = 0$ . Their definitions and numerical values

for  $\delta \neq 0$  are given in App. A. In the above flow equations, we have rescaled the couplings as  $(g^2, e^2)\Lambda^{-\epsilon}/(2\pi^2) \mapsto (1 - \delta^2)(g^2, e^2)$ , which turns out convenient to assess the properties of the stable fixed point. Note that this implies that higher-order loop corrections to the above equations will involve additional factors of  $(1 - \delta^2)$ . Importantly, in the present form, the stable fixed point associated with AIAO Weyl quantum criticality is located at finite couplings, as we show next.

Figure 2.5(a) depicts the fixed-point structure for  $N = 1$  on the critical hypersurface  $r = 0$  in the space spanned by the parameters  $\delta$ ,  $g^2$ , and  $e^2$ . Notably, both  $g^2$  and  $e^2$  are relevant couplings and flow towards a finite value in the infrared, Fig. 2.5(b). For finite  $g^2$ , the anisotropy parameter  $\delta$  does then no longer have a fixed point at  $\delta = 0$ , and instead flows towards maximal anisotropy  $\delta = -1$ , Fig. 2.5(c). Most importantly, there is a unique stable fixed point at finite couplings, located at  $\delta_\star = -1$  and  $g_\star^2 = e_\star^2 = 9\epsilon/(16N)$ . This fixed point, labeled as QCP in Figs. 2.5(a)–(c), corresponds to the continuous quantum phase transition between the symmetric Luttinger semimetal, described by the Luttinger-Abrikosov-Beneslavskii fixed point [281, 328–330] in the  $g = 0$  plane<sup>2</sup>, Fig. 2.5(d), and the AIAO-ordered Weyl semimetal. It is characterized by finite boson anomalous dimensions  $\eta_\phi = \eta_a = \epsilon$ , vanishing fermion anomalous dimension  $\eta_\psi = 0$ , and a dynamical critical exponent  $z = 2$ . From the flow of the tuning parameter  $r$ , we furthermore obtain the correlation-length exponent  $\nu = 1/(2 - \epsilon)$ . In Sec. 2.4.3, we show that particle-hole symmetry becomes emergent at the quantum critical point. Section 2.4.4 also contains a discussion of the other, unstable, fixed points.

The stable fixed point has remarkable properties. First of all, the fact that it is located at  $\delta_\star = -1$  implies that higher-loop corrections to the vertex renormalizations, Eqs. (2.8) and (2.9), vanish at criticality, see Sec. 2.4.5 for details. As a consequence, we find that our one-loop results  $z = 2$ ,  $\eta_\phi = \eta_a = 4 - d$ , and  $\nu = 1/(d - 2)$ , hold at all loop orders at the quantum critical point. The transition between Luttinger and AIAO Weyl semimetals in the pyrochlore iridates therefore realizes a rare instance of an interacting fermionic quantum critical point in a three-dimensional system that allows an exact determination of its critical properties. With the exact results at hand, we can resolve an apparent contradiction in the literature concerning the relevance or not of the cubic anisotropy at this quantum critical point [326, 327]. While the anisotropy parameter  $\delta$  vanishes at the Luttinger-Abrikosov-Beneslavskii fixed point, it flows towards  $\delta = -1$  at the quantum critical fixed point. Technically, the emergence of this maximal anisotropy at criticality arises from the order-parameter contributions to the fermion self-energy, which are neglected in the analysis of Ref. [327]. Apart from its striking universal features, the AIAO quantum critical fixed point also gives rise to an exotic quasiuniversal regime, characterized by approximate power laws with slowly drifting exponents, see Sec. 2.5 for further details.

<sup>2</sup>To be more precise, the fixed point denoted LAB in this chapter is merely a projection of the Luttinger-Abrikosov-Beneslavskii fixed point, located at  $r \rightarrow \infty$  [282], onto the critical hypersurface  $r = 0$ . This can also be seen from Table 3.1, for instance. As a projection of the infrared stable Luttinger-Abrikosov-Beneslavskii fixed point, this fixed point is allowed to possess (two) relevant directions, cf. Fig. 2.5(a,c).



**Figure 2.6.:** Feynman diagrams at the one-loop order contributing to (a,b) the fermion anomalous dimensions  $\eta_1$ ,  $\eta_\psi$  and the anisotropy parameter renormalization  $\Delta\delta$ , (c) the order-parameter anomalous dimension  $\eta_\phi$  and the tuning parameter renormalization  $\Delta r$ , (d) the Coulomb anomalous dimension  $\eta_a$ , and (e)–(h) the vertex renormalizations  $\Delta g$  and  $\Delta e$ , respectively. The contributions to the flow of  $e^2$  from (a) and (g), as well as those from (b) and (h), cancel as a consequence of a Ward identity. Figure reprinted from Ref. [301].

### 2.4.2. Derivation of flow equations

In this subsection, we provide some details on the derivation of the flow equations. In order to generalize our theory, Eqs. (2.1)–(2.3), to noninteger spatial dimensions  $2 < d < 4$ , we keep the general counting of dimensions in the couplings, but perform the angular integrations directly in the physical dimension  $d = 3$  [340]. This prescription is believed to be the most appropriate choice for the present type of models [331]. Integrating out fast modes with momenta  $q$  in the thin shell  $\Lambda/b < q < \Lambda$  and arbitrary frequencies  $\omega \in \mathbb{R}$  leads to the renormalized action for the slow modes as

$$\begin{aligned}
S_< = & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} \left\{ \sum_{i=1}^N \psi_i^\dagger \left[ b^{\eta_1} i\omega + b^{\eta_\psi} \sum_{a=1}^5 d_a(\vec{q}) \gamma_a + (\delta + \Delta\delta) \sum_{a=1}^5 s_a d_a(\vec{q}) \gamma_a \right] \psi_i \right. \\
& + \frac{1}{2} \phi \left[ b^{\eta_\phi} q^2 + (r + \Delta r) \right] \phi + \frac{1}{2} a b^{\eta_a} q^2 a \left. \right\} \\
& + \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{(2\pi)^2} \int_0^{\Lambda/b} \frac{d^d q_1 d^d q_2}{(2\pi)^{2d}} \left[ (g + \Delta g) \phi \sum_{i=1}^N \psi_i^\dagger \gamma_{45} \psi_i + i(e + \Delta e) a \sum_{i=1}^N \psi_i^\dagger \psi_i \right]. \quad (2.10)
\end{aligned}$$

Figure 2.6 shows the pertinent diagrams at the one-loop order, giving rise to the anomalous dimensions  $\eta_1$ ,  $\eta_\psi$ ,  $\eta_\phi$ ,  $\eta_a$ , the explicit anisotropy parameter renormalization  $\Delta\delta$ , and the vertex renormalizations  $\Delta g$ ,  $\Delta e$ . We rescale all frequencies as  $b^z \omega \mapsto \omega$ , with dynamical exponent  $z = 2 + \eta_1 - \eta_\psi$ , and all momenta as  $b\vec{q} \mapsto \vec{q}$ . Further, we renormalize the fields according to  $b^{-(2+d+z-\eta_\psi)/2} \psi \mapsto \psi$ ,  $b^{-(2+d+z-\eta_\phi)/2} \phi \mapsto \phi$ , and  $b^{-(2+d+z-\eta_a)/2} a \mapsto a$ . The effective action then becomes of the same form as the original action, but with renormalized parameters  $\delta$ ,  $g$ ,  $e$ , and  $r$ . Performing the loop integration leads to the flow equations for  $r = 0$  as given in Eqs. (2.7)–(2.9) of the previous subsection.

In particular, the one-loop fermion self-energy diagrams in Figs. 2.6(a) and (b) are frequency independent, such that  $\eta_1 = 0$ . This leads to the scaling relation  $z = 2 - \eta_\psi$ , with

the fermion anomalous dimension

$$\eta_\psi = \frac{2}{15}(1 - \delta^2)[(1 - \delta)(g^2 + e^2)f_{1e} - (1 + \delta)(g^2 - e^2)f_{1t}], \quad (2.11)$$

where we have rescaled the couplings as in Eqs. (2.7)–(2.9) of the previous subsection. Since there appears to be no fundamental reason for the cancellation of frequency dependencies in the fermion self-energy diagrams in general, we should expect the above scaling relation to receive corrections at higher loop orders. At the quantum critical point at  $\delta_\star = -1$ , however, these higher-loop corrections vanish, cf. Sec. 2.4.5.

By contrast, the self-energy contributions in Figs. 2.6(a) and (b) to the flow of  $e^2$  cancel with the explicit charge vertex renormalizations in Figs. 2.6(g) and (h), respectively, as a consequence of the Ward identity associated with the gauge transformation  $\psi \mapsto e^{ie\lambda(\tau)}\psi$ ,  $a \mapsto a - \partial_\tau \lambda(\tau)$  [282]. The flow of the effective charge can therefore be written as

$$\frac{de^2}{d \ln b} = \left( \epsilon + z - 2 - \eta_a + \frac{2\delta}{1 - \delta^2} \frac{d\delta}{d \ln b} \right) e^2, \quad (2.12)$$

where the last term arises from the reparametrization of the couplings as given below Eqs. (2.7)–(2.9) in the previous subsection. Equation (2.12) leads to the scaling relation  $\eta_a = \epsilon + z - 2$  at any charged fixed point in the Luttinger fermion system. As this scaling relation ultimately arises from a Ward identity, which holds at all loop orders within the  $\epsilon$  expansion, we expect it to be exact also at a quantum critical point away from  $\delta_\star = -1$ .

Finally, the flow equation for the tuning parameter  $r$  reads

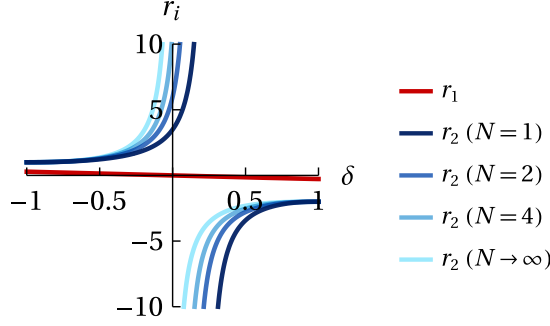
$$\frac{dr}{d \ln b} = (2 - \eta_\phi)r - \frac{4N}{5}(1 - \delta)(1 - \delta^2)g^2 f_{2e}, \quad (2.13)$$

where we have rescaled  $r\Lambda^{-2} \mapsto r$ , and  $g^2$  and  $e^2$  as in Eqs. (2.7)–(2.9) of the previous subsection. At the critical point, it determines the correlation-length exponent, leading to another scaling relation  $1/\nu = 2 - \eta_\phi$ . This scaling relation is basically a consequence of the fact that all bosonic self-interactions compatible with the symmetries of the model, such as a  $\phi^4$  term, are irrelevant in the renormalization group sense. We therefore expect it also to hold to all loop orders.

### 2.4.3. Emergent particle-hole symmetry

In this subsection, we demonstrate that particle-hole symmetry is emergent at the quantum critical point. To this end, we add to the Lagrangian the perturbation  $\mathcal{L}_x = -x \sum_{i=1}^N \psi_i^\dagger \nabla^2 \psi_i$ , with small parameter  $x$ ,  $|x| \ll 1$ . For  $x \neq 0$ ,  $\mathcal{L}_x$  breaks particle-hole symmetry explicitly [327]. The corresponding flow equation reads

$$\frac{dx}{d \ln b} = -\eta_\psi x, \quad (2.14)$$



**Figure 2.7.:** Right-hand sides of Eqs. (2.15) and (2.16) as a function of  $\delta$ , showing that  $r_1 \neq r_2$  for all values of  $\delta \in [-1, 1]$ , preventing the existence of any interacting fixed point at  $\delta \neq \pm 1$  for an arbitrary number of band touching points  $N \in \mathbb{N}$ . Figure reprinted from Ref. [301].

with  $\eta_\psi$  given in Eq. (2.11). Importantly,  $\eta_\psi > 0$  ( $\eta_\psi = 0$ ) for  $g^2 > 0$ ,  $e^2 > 0$ , and  $\delta \in (-1, 0]$  ( $\delta = -1$ ), corresponding to an irrelevant (marginal) particle-hole asymmetry parameter  $x$ . Assuming  $\delta < 0$  for the pyrochlore iridates [311],  $x$  flows to zero towards the infrared, and particle-hole symmetry becomes emergent in the quasiuniversal intermediate-temperature regime. Note that  $x$  is exactly marginal at the stable quantum critical fixed point. There is therefore a line of fixed points at  $\delta = -1$  within a finite interval around  $x = 0$ . We have verified, however, that the flow for any small perturbation  $x \neq 0$  and  $\delta - \delta_\star > 0$  is, in the ultra-low-energy limit, always towards the particle-hole-symmetric quantum critical fixed point at  $x = 0$ .

#### 2.4.4. Fixed-point structure

In this subsection, we give more details of the fixed-point structure on the critical hypersurface  $r = 0$ , including also the unstable fixed points with infrared relevant directions.

First of all, vanishing couplings  $g^2 = e^2 = 0$  lead to zero renormalization group flow for arbitrary values of the anisotropy parameter  $\delta \in [-1, 1]$ , leading to a line of Gaussian fixed points (see thick black line in Fig. 2.5(a)).

For  $g^2 = 0$ , but  $e^2 > 0$ , the flow equations host three interacting fixed points. One, located at  $\delta = 0$ , corresponds to the well-known Luttinger-Abrikosov-Beneslavskii fixed point [281, 328–330]. It is stable within the plane  $g^2 = 0$  (see Fig. 2.5(d)), but unstable in the direction perpendicular to it, as long as  $r$  is tuned to criticality (see Fig. 2.5(c)). Two further unstable fixed points are located at  $\delta = \pm 1$  and  $g^2 = 0$ , both of which have previously been encountered [327], but are of no great importance to this work, since not only are they unstable within the  $g^2 = 0$  plane, but also in the direction perpendicular to it.

For  $e^2 = 0$ , but  $g^2 > 0$ , we find an additional interacting fixed point, corresponding to AIAO quantum criticality in the absence of the long-range Coulomb repulsion, labeled as QCP<sub>0</sub> in Figs. 2.5(a,b). It is stable within the  $e^2 = 0$  plane, but unstable in the direction perpendicular to it.

**Table 2.1.:** Fixed points, their locations, and number of relevant (rel.) directions on the critical hypersurface  $r = 0$ .

| Label            | $g^2 N/\epsilon$ | $e^2 N/\epsilon$ | $\delta$      | Rel. directions | Comment                                                                      |
|------------------|------------------|------------------|---------------|-----------------|------------------------------------------------------------------------------|
| G                | 0                | 0                | $\in [-1, 1]$ | 2               | Line of Gaussian fixed points                                                |
| A                | 0                | 9/16             | -1            | 2               | Unstable fixed point at $\delta = -1$ ,<br>observed in [327]                 |
| LAB              | 0                | 15/19            | 0             | 1               | Luttinger-Abrikosov-Beneslavskii<br>fixed point, discussed in [281, 328–330] |
| A'               | 0                | 3/4              | 1             | 2               | Unstable fixed point at $\delta = 1$ ,<br>observed in [327]                  |
| QCP <sub>0</sub> | 9/16             | 0                | -1            | 1               | Quantum critical fixed point for<br>uncharged theory $e^2 = 0$               |
| QCP              | 9/16             | 9/16             | -1            | 0               | Unique stable fixed point for AIAO<br>Weyl quantum criticality               |

There is thus a unique stable fixed point on the critical hypersurface  $r = 0$ , which is the quantum critical point discussed in Sec. 2.4.1, located at  $(\delta_\star, g_\star^2, e_\star^2) = (-1, 9\epsilon/(16N), 9\epsilon/(16N))$ , and labeled as QCP in Figs. 2.5(a)–(c).

The fact that no further interacting real fixed point exists at some other value of  $\delta > -1$  on the critical hypersurface can also be seen as follows. Assume such a fixed point existed. The fact that  $f_{g^2}(\delta = 1) < 0$  implies that such additional fixed point would need to be located at  $\delta < 1$ , since any real fixed point at  $\delta = 1$  is prevented by the fixed-point condition  $dg^2/(d \ln b) = 0$  as long as  $g^2 > 0$ , see Eq. (2.8). For  $\delta \neq \pm 1$ , however, the fixed-point condition  $d\delta/(d \ln b) = 0$  can only be satisfied if

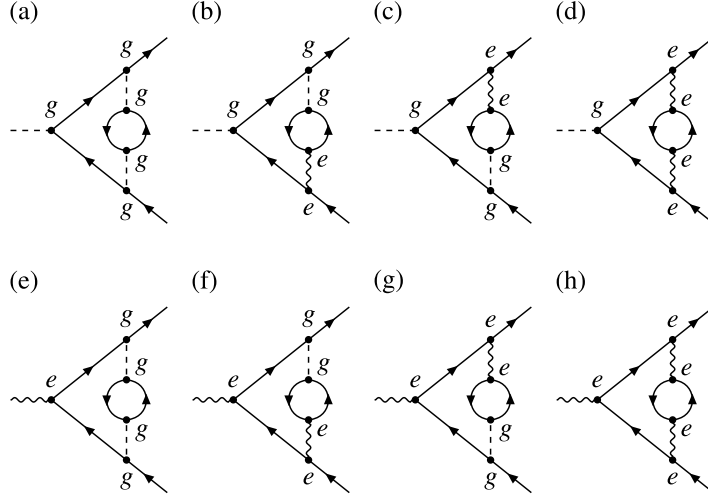
$$\frac{g^2}{e^2} = \frac{f_{1t}(\delta) - f_{1e}(\delta)}{f_{1t}(\delta) + f_{1e}(\delta)} =: r_1(\delta). \quad (2.15)$$

A second relation, which simultaneously has to hold at the putative fixed point, arises from the condition that the coupling ratio  $(g^2/e^2)$  does not flow,  $d(g^2/e^2)/(d \ln b) = 0$ , which can be simplified to

$$\frac{g^2}{e^2} = \frac{2(1-\delta)^2(1+\delta)f_{2e}(\delta) + 5Nf_{e^2}(\delta)}{2(1-\delta)^2(1+\delta)f_{2e}(\delta) + 5Nf_{g^2}(\delta)} =: r_2(\delta). \quad (2.16)$$

The right-hand sides of Eqs. (2.15) and (2.16) define functions  $r_i$ ,  $i = 1, 2$ . Figure 2.7 shows that  $r_1(\delta) \neq r_2(\delta)$  for all values of  $\delta \in [-1, 1]$  and any number of quadratic band touching points  $N \in \mathbb{N}$ . This implies that the two fixed-point conditions for a putative interacting fixed point located at  $\delta \neq \pm 1$ , Eqs. (2.15) and (2.16), cannot be simultaneously satisfied. At the one-loop order, there are therefore no other interacting fixed points; in particular, the stable quantum critical fixed point discussed in the main text is unique.

All fixed points on the critical hypersurface  $r = 0$ , their locations, and number of infrared relevant directions, are summarized in Table 2.1.



**Figure 2.8.:** Vertex corrections with largest possible number of closed fermion loops at two-loop order, contributing to the renormalization of (a)–(d) the Yukawa coupling  $g$  and (e)–(h) the effective charge  $e$ . Figure reprinted from Ref. [301].

### 2.4.5. Higher-loop corrections

In this subsection, we substantiate our claim that the critical behavior at the antiferromagnetic Weyl quantum critical point is one-loop exact. In particular, we argue that higher-loop corrections to the critical exponents vanish when  $\delta$  approaches the fixed-point value  $\delta_\star = -1$ . We first note any closed fermion loop in a given diagram leads to a pole  $\propto 1/(1-\delta^2)$ , while any loop containing at least one inner boson line leads to only a regular contribution as a function of  $\delta$ . This implies that the only diagrams that contribute, at a given loop order, to the critical exponents of a quantum critical point at  $\delta_\star = -1$  are those that involve the largest possible number of closed fermion loops.

To illustrate this point, let us consider the two-loop vertex corrections to the flows of the Yukawa coupling and the effective charge. The two-loop diagrams with the largest number of closed fermion loops contributing to the explicit vertex corrections are given in Fig. 2.8. These diagrams are most conveniently evaluated using dimensional regularization. We start with the correction  $\Gamma_\phi^{(2)} \propto g^5$  given in Fig. 2.8(a), which contributes to the flow of the Yukawa coupling. The evaluation of this diagram involves the one-loop order-parameter self-energy at arbitrary external frequencies and momenta,

$$\Sigma_\phi^{(1)}(\omega, \vec{p}) = 2Ng^2 \int \frac{d^d q}{(2\pi)^d} \frac{(Q_-^2 + Q_+^2)(Q_\times^4 - Q_-^2 Q_+^2)}{Q_-^2 Q_+^2 [\omega^2 + (Q_-^2 + Q_+^2)^2]}. \quad (2.17)$$

In the above equation, we have already performed the frequency integral, and have used the abbreviations  $Q_\pm^4 \equiv \sum_a (1 + s_a \delta)^2 d_a^2 (\vec{q} \pm \frac{\vec{p}}{2})$  and  $Q_\times^4 \equiv \sum_a s_a (1 + s_a \delta)^2 d_a (\vec{q} - \frac{\vec{p}}{2}) d_a (\vec{q} + \frac{\vec{p}}{2})$ . In  $d = 4 - \epsilon$  spatial dimensions, the substitution  $\vec{q} \mapsto \sqrt{|\omega|} \vec{q}$  reveals the scaling form for the boson self-energy,  $\Sigma_\phi^{(1)}(\omega, \vec{p}) = Ng^2 |\omega|^{1-\frac{\epsilon}{2}} \mathcal{F}_{\epsilon, \delta}(\vec{p}/\sqrt{|\omega|})$  with scaling function  $\mathcal{F}_{\epsilon, \delta}$ . It follows that there is a unique divergent term in the sense of dimensional regularization,

which is frequency independent,

$$\Sigma_\phi^{(1)}(\omega, \vec{p}) - \Sigma_\phi^{(1)}(0, 0) = Ng^2 \frac{f_{g^2}(\delta)}{1 - \delta^2} \frac{p^2}{\epsilon} + \text{finite terms}, \quad (2.18)$$

displaying the advertised pole  $\propto 1/(1 - \delta^2)$  as a function of  $\delta$ . The two-loop correction  $\Gamma_\phi^{(2)}$  given in Fig. 2.8(a) now can be reduced to an expression involving only the one-loop correction  $\Gamma_\phi^{(1)}$  given in Fig. 2.6(e), of the form

$$\Gamma_\phi^{(2)} \propto Ng^2 \frac{f_{g^2}(\delta)}{1 - \delta^2} \Gamma_\phi^{(1)}, \quad (2.19)$$

with  $\delta$ -independent prefactors. We observe that the rescaling of the couplings given below Eqs. (2.7)–(2.9) in the main text precisely cancels the  $1/(1 - \delta^2)$  divergence caused by the internal fermion loop, such that the two-loop contribution to the flow of the Yukawa coupling reads

$$\left. \frac{dg^2}{d \ln b} \right|_{\text{Fig. 2.8(a)}} = \frac{2}{5} N(1 - \delta)(1 - \delta^2) f_{2e}(\delta) f_{g^2}(\delta) g^6. \quad (2.20)$$

Importantly, for  $\delta = -1$ , the two-loop correction is suppressed. Similar scaling relations and reduction formulas, relating two-loop vertex corrections with one-loop boson self-energies and vertex corrections, can be derived for all remaining diagrams shown in Figs. 2.8(b)–(h). In particular, the two-loop diagram depicted in Fig. 2.8(d) gives rise to a completely analogous expression as displayed in Eq. (2.20). Moreover, the two-loop contributions in Figs. 2.8(e,h) turn out to vanish exactly, since the corresponding one-loop diagrams are already zero. Also, the four remaining two-loop vertex corrections given in Figs. 2.8(b,c,f,g) are exactly zero, since the closed fermion loop connected to a photon field and an order-parameter field vanishes by symmetry. In sum, the flows of the Yukawa coupling and the effective charge can be written at the two-loop order as

$$\frac{dg^2}{d \ln b} = (\epsilon - \eta_\phi) g^2 + \mathcal{O}(\delta - \delta_\star), \quad (2.21)$$

$$\frac{de^2}{d \ln b} = (\epsilon - \eta_a) e^2 + \mathcal{O}(\delta - \delta_\star), \quad (2.22)$$

which implies  $\eta_\phi = \epsilon + \mathcal{O}(\epsilon^3)$  and  $\eta_a = \epsilon + \mathcal{O}(\epsilon^3)$  at the quantum critical point. Analogously, from the flow of the tuning parameter  $r$ , we can show that  $1/\nu = 2 - \epsilon + \mathcal{O}(\epsilon^3)$  at the two-loop order. In fact, analogous arguments apply for any given loop order, since the only diagrams that contribute at  $\delta = -1$  are those that involve the largest possible number of closed fermion loops, implying that they can be reduced to products of one-loop self-energies and vertex corrections. These, however, will always be suppressed at  $\delta = -1$ , implying that  $\eta_\phi = 4 - d$ ,  $\eta_a = 4 - d$ ,  $z = 2$ , and  $1/\nu = d - 2$  exactly at the quantum critical point, as stated in Sec. 2.4.1. We emphasize that while these results have been obtained within a particular dimensional continuation scheme, the above analysis of the higher-loop corrections can be carried out in an analogous way (although somewhat less explicitly), without ever specifying the scheme. Our finding that the flow near the critical fixed point

is one-loop exact and higher-loop corrections vanish at the quantum critical point, and the resulting critical behavior, is therefore independent of our particular approach, and continues to hold for other dimensional continuation schemes.

## 2.5. Quasiuniversality

Having examined the genuinely universal ultra-low-energy regime in the preceding sections, let us now turn to larger energy scales and explore an exotic quasiuniversal intermediate-to-low-energy regime, cf. Fig. 2.1. First, we introduce and exemplify a novel mechanism for such quasiuniversal behavior, see Sec. 2.5.1. Subsequently, the quasiuniversal behavior encountered is contrasted with the universal behavior near a generic quantum critical fixed point, see Sec. 2.5.2.

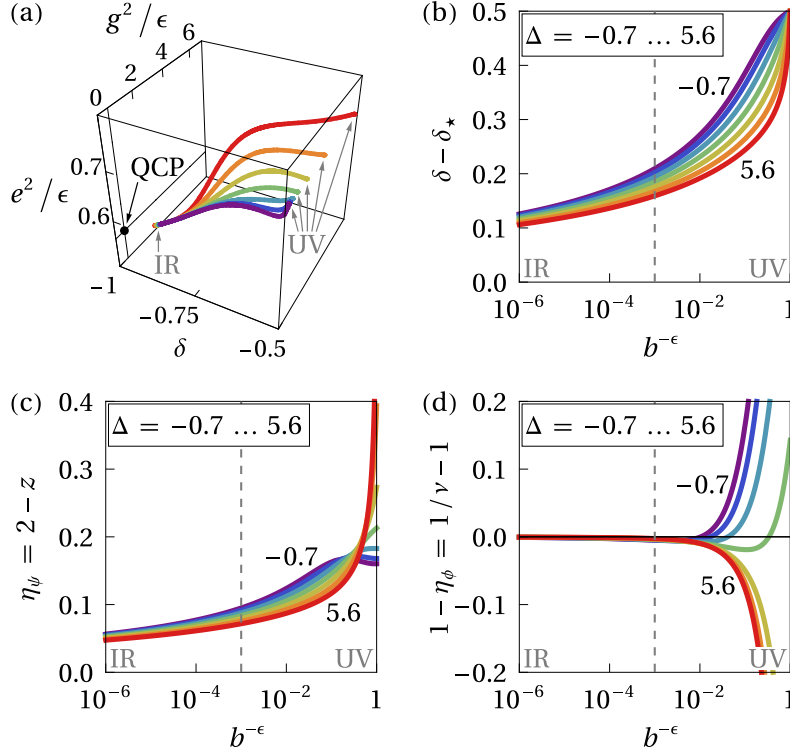
### 2.5.1. Quasiuniversality from marginally irrelevant operator

In the vicinity of the quantum critical fixed point at  $\delta_\star = -1$ , the flow of the anisotropy parameter takes the form

$$\left. \frac{d(\delta - \delta_\star)}{d \ln b} \right|_{g_\star^2, e_\star^2} = -\frac{c\epsilon}{N}(\delta - \delta_\star)^2 + \mathcal{O}((\delta - \delta_\star)^3) \quad (2.23)$$

with a constant  $c := 3f_{1e}(-1)/5 \approx 0.4449$ . Note the absence of a linear term  $\propto (\delta - \delta_\star)$  in the above equation. The deviation  $(\delta - \delta_\star)$  corresponds to a *marginally* irrelevant parameter, implying a logarithmically slow flow towards the critical point,  $\delta(b) - \delta_\star \simeq N/(c\epsilon \ln b)$ , for  $\ln b \gg 1$ . By contrast, the Yukawa coupling and the effective charge acquire a power-law flow,  $g^2(b) - g_\star^2 \propto b^{-\epsilon}$  and  $e^2(b) - e_\star^2 \propto b^{-\epsilon}$ . This implies a separation of scales, giving rise to the following three regimes in energy. In the nonuniversal high-energy regime, the couplings  $g^2$  and  $e^2$  flow from their microscopic values towards pseudo-fixed-point values  $g_\star^2(\delta)$  and  $e_\star^2(\delta)$ , which depend only on the anisotropy parameter  $\delta$ . Within this regime, the latter can be considered approximately constant. Its slow flow becomes visible only when several orders of magnitude in energy are considered. This defines a quasiuniversal intermediate-to-low-energy regime [233, 275], in which the couplings  $g^2$  and  $e^2$  do no longer depend on their microscopic values, but solely follow their pseudo-fixed-point values  $g_\star^2(\delta)$  and  $e_\star^2(\delta)$  with slowly varying anisotropy parameter  $\delta$ . In this regime, the correlation length becomes large, such that observables display approximate power laws with slowly drifting exponents [341, 342]. Finally, the genuinely universal ultra-low-energy regime will be reached only after the anisotropy parameter has approached its ultimate infrared regime, which requires fluctuations on unusually many energy scales to be integrated out.

This behavior is exemplified in Fig. 2.9, which shows the numerically integrated renormalization group flow for different initial couplings on the critical hypersurface  $r = 0$ .



**Figure 2.9.:** Renormalization group flow on the critical hypersurface  $r = 0$  from the ultraviolet scale, corresponding to  $b^{-\epsilon} = 1$ , to a deep infrared scale, corresponding to  $b^{-\epsilon} = 10^{-6}$ , starting from different microscopic parameters. Each curve corresponds to a numerical integration of the flow equations using the initial conditions  $(\delta, g^2, e^2) = (-0.5, 0.7391, 0.9098 + \Delta)$  for  $b^{-\epsilon} = 1$ , with  $\Delta = -0.7, -0.6, -0.4, 0, 0.8, 2.4, 5.6$  from purple to red. The starting values of  $(g^2, e^2)$  have been chosen to satisfy the pseudo-fixed-point conditions for  $\delta = -0.5$  when  $\Delta = 0$ . (a) Flow trajectories in the parameter space spanned by  $\delta$ ,  $g^2$ , and  $e^2$ , illustrating the crossover from the nonuniversal regime, characterized by independent trajectories, to the quasiuniversal regime, in which these collapse onto a single curve. Deviations of (b) anisotropy parameter  $\delta$  and effective exponents (c)  $\eta_\psi = 2 - z$  and (d)  $\eta_\phi = 2 - 1/\nu$  from their respective critical values as a function of renormalization group scale  $b^{-\epsilon}$ . The quasiuniversal regime emerges at  $b^{-\epsilon} \lesssim 10^{-3}$ , as indicated by the dashed line, and manifests itself in  $(g^2, e^2)$ -independent, but anisotropy-dependent, drifting exponents. Note that the flow is still significantly away from its ultra-low-energy limit even at  $b^{-\epsilon} = 10^{-6}$ , as evidenced by the finite deviation  $\delta - \delta_* \gtrsim 0.1$  in (b). Figure reprinted from Ref. [301].

The couplings  $g^2$  and  $e^2$  first exhibit a fast flow on individual, nonuniversal, trajectories, but then approach a single, quasiuniversal, trajectory of pseudo-fixed points  $(g_*^2(\delta), e_*^2(\delta))$ , along which they only slowly flow, as a consequence of the slow flow of  $\delta$ , Fig. 2.9(a). In fact, even after six orders of magnitude in  $b^\epsilon$  have been integrated out,  $\delta$  is still significantly away from its ultra-low-energy value  $\delta_* = -1$ , Fig. 2.9(b). There is therefore a large quasiuniversal regime characterized by approximate power laws with effective critical exponents, which are only slowly drifting with energy or length. As shown in Sec. 2.4.2, the present model satisfies additional scaling relations between the different exponent, viz.  $z = 2 - \eta_\psi$  and  $1/\nu = 2 - \eta_\phi$ . There are therefore only two independent effective exponents. Their slow drifts as a function of scale are depicted in Figs. 2.9(c) and (d), illustrating the fact that the genuinely universal values are reached, at least for some of the exponents, only in the ultra-low-energy limit. We emphasize that the slow renormalization group flow is independent of microscopic parameters and as such will occur in any

given material realization of the all-in-all-out Weyl quantum critical point in Luttinger semimetals. In Sec. 2.5.2, we compare the quasiuniversal flow with a generic power-law flow of a toy model that does not feature a marginal coupling, illustrating the unusual behavior observed in the quasiuniversal regime.

We emphasize that quasiuniversal behavior will emerge near any quantum critical fixed point that possesses exactly one marginally irrelevant coupling and is located on a critical manifold of dimension two or higher. In contrast to quasiuniversality arising either from a fixed-point annihilation or from a flat region of the beta function, cf. Sec. 1.4.3, quasiuniversality discussed here originates from the very presence of a quantum critical fixed point. In other words, the mechanism proposed in this chapter offers a scenario in which quasiuniversal behavior at intermediate-to-low-energy scales and universal behavior at the ultra-low-energy scale are not mutually exclusive, but both the result of the same genuine quantum critical fixed point. Moreover, the AIAO Weyl quantum critical point clearly goes beyond the Landau-Ginzburg-Wilson paradigm, as it involves gapless fermions and also a gauge field, cf. Sec. 1.3. However, a potential purely bosonic analogue of the AIAO Weyl quantum critical point, for which there is no fundamental reason to believe it does not exist, may still be viewed as a (quantum) critical point beyond the Landau-Ginzburg-Wilson paradigm in the strict sense, since a marginally irrelevant operator yields an exotic phenomenology and is not present in standard Landau-Ginzburg-Wilson theories, cf. Sec. 1.1. Ultimately, this is a matter of definition. Further, note that a marginally irrelevant operator  $\delta$  of higher “order”  $p > 2$ , where  $p$  denotes the number of vanishing terms in the Taylor expansion around a quantum critical fixed point, not only maintains the validity of the proposed mechanism, but also enlarges the quasiuniversal regime, as perturbations decay even more slowly,  $\delta(b) - \delta_\star \propto (\ln b)^{-1/(p-1)}$ , for  $\ln b \gg 1$ . Lastly, we refer the reader to Sec. 2.6 for an experimental perspective on the proposed hierarchy of energy scales.

### 2.5.2. Quasiuniversal flow vs. generic flow

In this subsection, we illustrate the unusual behavior observed in the quasiuniversal regime by comparing the quasiuniversal flow with a generic power-law flow of a toy model that does not feature a marginal coupling. To construct such a toy flow, we manually modify the flow equation of the anisotropy  $\delta$ , Eq. (2.7), in order to remove its “marginality.” To this end, we simply replace  $(1 + \delta)^2$  by  $(1 + \delta)$ , giving rise to the toy flow equation

$$\frac{d\delta}{d \ln b} = -\frac{2}{15}(1 - \delta)^2(1 + \delta) \left[ (g^2 + e^2)f_{1e} + (g^2 - e^2)f_{1t} \right]. \quad (2.24)$$

We leave the flow equations for the Yukawa coupling  $g^2$  and the effective charge  $e^2$  unchanged. As a consequence, the system still features an AIAO Weyl quantum critical point at  $\delta_\star = -1$ , characterized by the same leading universal critical exponents  $z = 2$ ,  $\eta_\psi = 0$ ,  $\eta_\phi = \eta_a = \epsilon$ , and  $1/\nu = 2 - \epsilon$ . However, the flow towards this fixed point in this toy model is qualitatively different, since the anisotropy  $\delta$  is no longer marginally irrelevant, but now

represents a standard power-law irrelevant parameter at the quantum critical point. This can be seen by expanding the above flow equation about the fixed point, reading

$$\left. \frac{d(\delta - \delta_\star)}{d \ln b} \right|_{g_\star^2, e_\star^2} = -\frac{c\epsilon}{N}(\delta - \delta_\star) + \mathcal{O}((\delta - \delta_\star)^2), \quad (2.25)$$

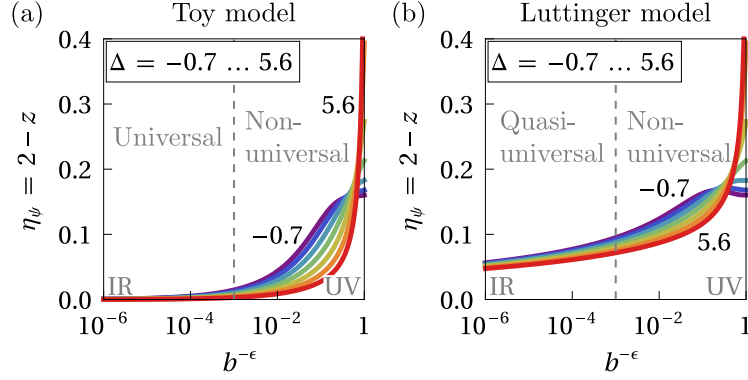
with the constant  $c > 0$  as given below Eq. (2.23) in the previous subsection. Importantly, the above flow equation now features a linear term  $\propto (\delta - \delta_\star)$ , indicating the absence of a marginal parameter in the toy flow. Consequently, the flow on the critical manifold  $r = 0$  is of usual power-law behavior towards the infrared stable fixed point. This qualitative change manifests itself also in all flowing exponents. They approach their true infrared values rapidly and no longer display intermediate drifting behavior. This is illustrated in Fig. 2.10, which shows the flowing fermion anomalous dimension  $\eta_\psi$  as a function of renormalization group scale  $b^{-\epsilon}$ . We reiterate that  $\eta_\psi$  is related to the dynamical critical exponent  $z$  via  $\eta_\psi = 2 - z$ , and as such governs the behavior of the specific heat  $C \propto T^{d/z}$ , cf. Sec. 2.6.1. In the toy-model flow,  $\eta_\psi$  approaches zero already at  $b^{-\epsilon} \simeq 10^{-3}$ , see Fig. 2.10(a). By contrast, the Luttinger-model flow features a wide quasiuniversal regime for  $b^{-\epsilon} \lesssim 10^{-3}$  up to significantly below  $b^{-\epsilon} = 10^{-6}$ , characterized by a slow drift of  $\eta_\psi = 2 - z$ , see Fig. 2.10(b). We have verified that all other flowing exponents display an analogous behavior.

## 2.6. Experimental aspects

Finally, let us return to the experimental motivation of this project, outlined in Sec. 2.1.2, and discuss the implications of our calculations, see Sec. 2.6.1. In addition to identifying clear experimental signatures of quasiuniversality, we present recent progress towards their first experimental observation, see Sec. 2.6.2. In spirit, this section also returns us to Kadanoff's quote at the very beginning of this chapter, an expression of both joy and astonishment at the experimental confirmation of a theory, a sentiment the author of this dissertation wholeheartedly shares.

### 2.6.1. Experimental signatures

The quasiuniversal regime is separated by crossovers from the nonuniversal regime at the ultraviolet scale and the genuinely universal regime in the deep infrared, and realizes a novel strongly interacting quantum state of matter. For the pyrochlore iridates, the relevant energy scale above which the physics depends on microscopic details of the material is around 100 K [303]. Assuming  $z \simeq 2$ , each order of magnitude in energy corresponds to only half an order of magnitude in the renormalization group momentum scale  $b$ . The ultra-low-temperature behavior will therefore be reached only well below the 1 mK regime, cf. Fig. 2.9. Such a regime is not only difficult to access experimentally and necessitates sufficiently pure crystals that do not cut off the required long-range fluctuations: in the

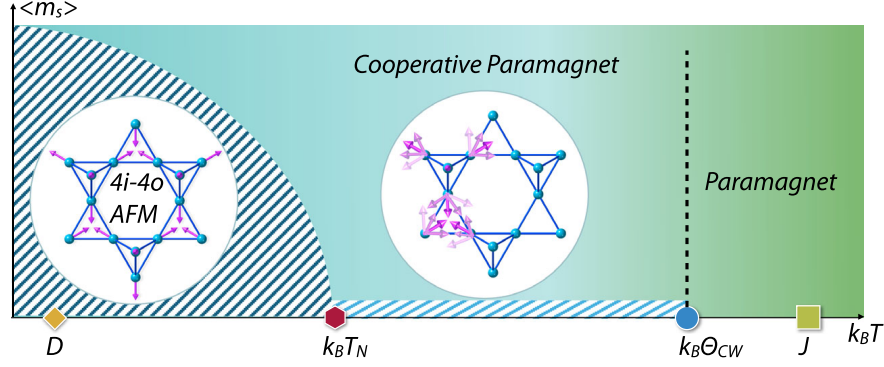


**Figure 2.10.:** Flowing fermion anomalous dimension  $\eta_\psi$  as a function of renormalization group scale  $b^{-\epsilon}$  from the ultraviolet scale, corresponding to  $b^{-\epsilon} = 1$ , to a deep infrared scale, corresponding to  $b^{-\epsilon} = 10^{-6}$ , starting from different microscopic parameters on the critical hypersurface  $r = 0$ . Each curve corresponds to a numerical integration of the flow equations using the initial conditions  $(\delta, g^2, e^2) = (-0.5, 0.7391, 0.9098 + \Delta)$  for  $b^{-\epsilon} = 1$ , with  $\Delta = -0.7, -0.6, -0.4, 0, 0.8, 2.4, 5.6$  from purple to red, as in Fig. 2.9. Panel (a) shows the toy-model flow, in which the universal regime emerges at  $b^{-\epsilon} \lesssim 10^{-3}$ , as indicated by the dashed line, which manifests itself in constant exponents that are independent of any microscopic parameters. Panel (b) shows the Luttinger-model flow for direct comparison (same as Fig. 2.9(c)), in which the quasiuniversal regime emerges at  $b^{-\epsilon} \lesssim 10^{-3}$ , as indicated by the dashed line, which manifests itself in slowly drifting exponents. Figure reprinted from Ref. [301].

case of the pyrochlore iridates, we also expect new effects in this ultra-low-energy regime, arising from the weak Kondo coupling of the iridium electrons to the rare-earth local moments [343–345]. The experimentally most easily accessible regime below 100 K, by contrast, will be governed by quasiuniversal behavior, characterized by approximate power laws with slowly drifting exponents. The specific heat as a function of temperature, for instance, is expected to scale as  $C \propto T^{d/z}$  with drifting exponent  $d/z > 3/2$  at around 100 K and  $d/z = 3/2$  in the ultra-low-temperature limit. Observing such drifting exponents, e.g., in careful thermodynamic or transport measurements, accomplishes an experimental realization of this new state of matter.

### 2.6.2. Experimental observation

In this subsection, we present recent experimental progress on the rare-earth pyrochlore-iridate family  $R_2\text{Ir}_2\text{O}_7$ , ultimately refining Fig. 2.3. In particular, we focus on new elastic and inelastic X-ray scattering studies on (111)-oriented  $\text{Y}_2\text{Ir}_2\text{O}_7$  thin films, supplemented with linear spin wave theory and density functional calculations, as outlined in Ref. [346]. Below the Néel temperature  $T_N \approx 145$  K, the authors report an AIAO-ordered antiferromagnetic Weyl phase, in agreement with the theoretical results outlined above. Upon raising the temperature,  $\text{Y}_2\text{Ir}_2\text{O}_7$  undergoes a thermal 3d Ising phase transition into the paramagnetic phase [346], once again consistent with theoretical expectations. Surprisingly, spectrally sharp, i.e., long-lived, gapped magnon modes, typically associated with magnetically ordered phases, persist beyond the Néel temperature  $T_N$  and up to a crossover scale  $T_\star$  [346], interpreted as a cooperative paramagnet [347–352]. Typically, cooperative paramagnetism has been discussed in the context of insulators, with spin ice serving as a

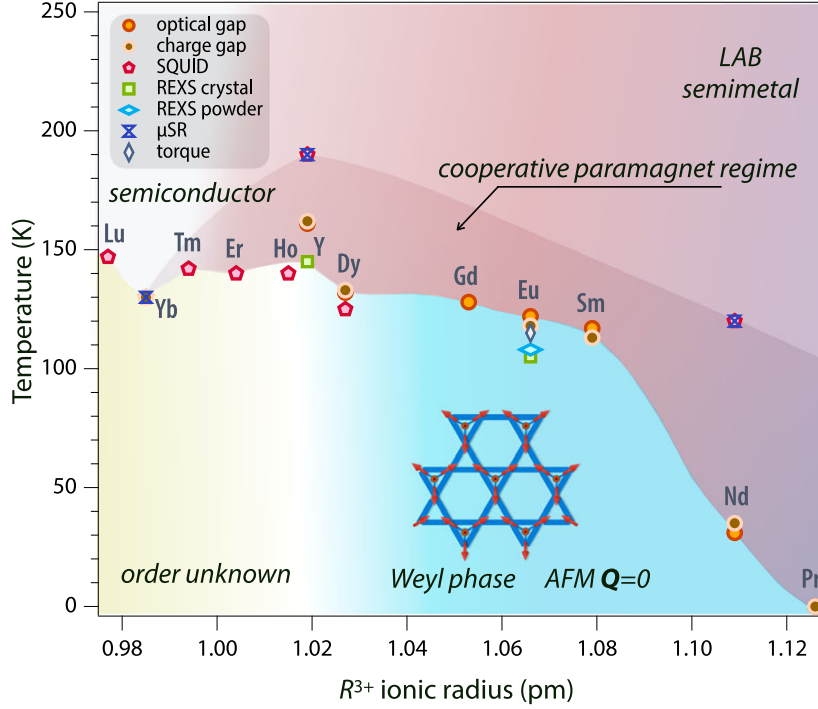


**Figure 2.11.:** Staggered magnetization  $\langle m_s \rangle$  schematically shown as a function of the thermal energy  $k_B T$  for  $Y_2Ir_2O_7$ . The AIAO-ordered antiferromagnetic (4i-4o AFM) phase persists up to the Néel temperature  $T_N \approx 145$  K, above which the paramagnetic phase sets in. The latter consists of two regimes: an unconventional cooperative paramagnetic regime presumably below the Curie-Weiss temperature  $350 \text{ K} \lesssim |\Theta_{CW}| \lesssim 450 \text{ K}$  and the ordinary paramagnetic regime above  $|\Theta_{CW}|$ , separated by a crossover (dashed line). The figure, reprinted from Ref. [346], also displays the Dzyaloshinskii-Moriya interaction  $D$  and the Heisenberg coupling  $J$  obtained from model fitting, which, however, are not the focus of our discussion here.

prototypical example [353]. The findings presented in Ref. [346] may thus be understood as the first experimental observation of a cooperative paramagnetic regime in a metallic environment in three spatial dimensions. Within the approach of Ref. [346], the crossover temperature is estimated to be at least  $T_* \gtrsim 175$  K, but is expected to be of the same order as the Curie-Weiss temperature  $350 \text{ K} \lesssim |\Theta_{CW}| \lesssim 450 \text{ K}$  [354, 355] (and the Heisenberg coupling  $J$ ), since both scales mark the crossover from a correlated paramagnetic regime at intermediate temperatures to an uncorrelated paramagnetic regime at high temperatures. This scenario is depicted in Fig. 2.11. The existence of a cooperative paramagnetic regime is further supported by the observation that the magnetic entropy is significantly lower than expected for an ordinary paramagnetic state [356].

Combining these new insights from X-ray scattering on  $Y_2Ir_2O_7$  thin films [346] with previous findings from optical and charge gap [357], superconducting quantum interference device (SQUID) [358–361], X-ray scattering [362–364], torque magnetometry [365], and muon spin resonance ( $\mu$ SR) [366, 367] measurements on the whole rare-earth pyrochlore-iridate family  $R_2Ir_2O_7$ , results in a refined version of the finite-temperature phase diagram introduced in the motivation (Fig. 2.3), depicted in Fig. 2.12. The broad cooperative paramagnetic regime has been argued to coincide with the predicted quasiuniversal regime<sup>3</sup>, for instance, since the latter also hosts strong but short-range spin-spin correlations [346]. This can be seen from the on-shell AIAO susceptibility  $\chi$ , expected to scale as  $\chi \propto |\omega|^{-1}$  in the noninteracting paramagnetic regime at high temperatures and as  $\chi \propto |\omega|^{-(2-\eta_\phi)/z}$  in the interacting quasiuniversal regime at intermediate temperatures, with, in general, scale-dependent anomalous dimension  $\eta_\phi$  and scale-dependent dynamical exponent  $z$ . To further strengthen the connection between the cooperative paramagnetic regime and the

<sup>3</sup>We note that while both regimes share similarities (such as strong short-range spin-spin correlations), the experimental observation of *sharp* magnon modes is at odds with the expected continuum of excitations in the quasiuniversal regime, cf. Fig. 1.2. This requires further investigation before reaching a definite conclusion.



**Figure 2.12.:** State-of-the-art finite-temperature phase diagram as a function of the  $R^{3+}$  ionic radius for the rare-earth pyrochlore-iridate family  $R_2\text{Ir}_2\text{O}_7$ . As indicated in the inset, the phase diagram is deduced from optical and charge gap measurements [357], magnetization measurements (SQUID) [358–361], X-ray scattering [346, 362–364], torque magnetometry [365], and  $\mu\text{SR}$  [366, 367]. Larger pyrochlore iridates, most probably except  $\text{Pr}_2\text{Ir}_2\text{O}_7$ , realize an AIAO-ordered anti-ferromagnetic Weyl state at low temperatures. Above the corresponding Néel temperatures  $T_N$ , a paramagnetic phase with unusually strong but short-range spin-spin correlations emerges, which has been argued to arise as a consequence of the marginally irrelevant spatial anisotropy at the putative quantum critical point near  $\text{Pr}_2\text{Ir}_2\text{O}_7$ . The thermal phase transition is continuous and falls into the 3d Ising universality class. The cooperative paramagnetic regime at intermediate temperatures turns into an uncorrelated paramagnetic regime above a crossover scale  $T_*$ . We note that the figure, reprinted from Ref. [346], displays the  $R^{3+}$  ionic radius in angstroms rather than in picometers.

quasiuniversal regime, the observation of drifting exponents in thermodynamic or transport quantities within the cooperative paramagnetic regime would provide strong additional evidence. This constitutes an excellent direction for future experimental research and, in case of a positive outcome, would mark the first definite observation of a quasiuniversal regime. We emphasize that the exotic behavior of the rare-earth pyrochlore iridates is of great interest not only for fundamental research but also for technological applications [346, 368].

## 2.7. Conclusion

In this chapter, we have proposed and exemplified a novel mechanism for quasiuniversality that relies on the presence of a marginally irrelevant operator at the quantum critical fixed point. The proposed mechanism thus offers a scenario in which quasiuniversal behavior at intermediate-to-low-energy scales and genuine universal behavior at the ultra-low-

energy scale are not contradictory, but rather both consequences of an exotic quantum critical point. This should be contrasted with fixed-point annihilation scenarios [274, 275], which also exhibit quasiuniversal behavior at intermediate-to-low-energy scales, but display generic running-coupling behavior at lower scales, cf. Sec. 1.4.3. We have demonstrated the emergence of such a novel quasiuniversal regime in the finite-temperature phase diagram of interacting three-dimensional Luttinger semimetals. Experimental signatures have been proposed, and recent experimental findings discussed, with promising prospects for observing a quasiuniversal regime in the finite-temperature phase diagram of the rare-earth pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$  [346]. Moreover, the universal regime of the AIAO Weyl quantum critical point proves to be equally compelling. Within a renormalization group analysis, we have determined all critical exponents fully analytically and established their one-loop exact structure. As a side product, we have demonstrated the emergence of maximal cubic anisotropy at criticality, thereby resolving an apparent contradiction in the literature [326, 327].

There are numerous excellent directions for future research, here we highlight a few of them. On the experimental side, an important task is to further establish the connection between the cooperative paramagnetic regime and the quasiuniversal regime of the pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ , for instance by observing drifting exponents in thermodynamic or transport quantities in the cooperative paramagnetic regime, as outlined in Sec. 2.6.1. On the theoretical side, it would be interesting to cross-check the critical exponents of the AIAO Weyl quantum critical point using complementary approaches, such as large- $N$  techniques or quantum Monte Carlo simulations, and verify their exactness. An early-stage collaboration on SLAC-fermion quantum Monte Carlo simulations of three-dimensional Luttinger fermions is underway. Further, the search for even more exotic physics in the ultra-low-energy regime, likely induced by the weak Kondo coupling between the iridium electrons and the rare-earth local moments, represents another excellent direction for future theoretical work [345]. From a phenomenological perspective, a natural direction concerns the study of (quantum) critical points involving different kinds of marginal operators. This includes critical theories with marginally relevant tuning parameters, potentially leading to unusually soft second-order phase transitions, and critical theories with multiple marginally irrelevant couplings, potentially generating an even richer hierarchy of scales than encountered in this chapter. From a high-energy perspective as well, scenarios involving marginal operators may help to overcome or reduce the hierarchy problem by naturally introducing a hierarchy of scales [369]. On a conceptual level, one may ask whether the quantum critical behavior of a system with an extended Fermi surface can be understood as the large- $N$  solution of an interacting semimetal theory at maximal spatial anisotropy, say  $\delta_\star = -1$ , cf. Sec. 2.4. The resulting flat direction in the fermionic spectrum may be used to reconstruct the extended Fermi surface, potentially yielding an alternative to traditional patch theories [46].



### 3. Continuous order-to-order phase transition from fixed-point annihilation

*“Any good theorem should have several proofs, the more the better.” [370]*

— Michael F. Atiyah

A central concept in the theory of phase transitions beyond the Landau-Ginzburg-Wilson paradigm is fractionalization: the formation of new quasiparticles that interact via emergent gauge fields. This concept has been extensively explored in the context of continuous quantum phase transitions between distinct orders that break different symmetries. In this chapter, we propose a mechanism for continuous order-to-order quantum phase transitions that operates independently of fractionalization. This mechanism is based on the collision and annihilation of two renormalization group fixed points: a quantum critical fixed point and an infrared stable fixed point. The annihilation of these fixed points rearranges the flow topology, eliminating the disordered phase associated with the infrared stable fixed point and promoting a second critical fixed point, unaffected by the collision, to a quantum critical point between distinct orders. We argue that this mechanism is relevant to a broad spectrum of physical systems. In particular, it can manifest in Luttinger fermion systems in three spatial dimensions, leading to a continuous quantum phase transition between an antiferromagnetic Weyl semimetal state, which breaks time-reversal symmetry, and a nematic topological insulator, characterized by broken lattice rotational symmetry. This continuous antiferromagnetic-Weyl-to-nematic-insulator transition might be observed in rare-earth pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ . Other possible realizations include kagome quantum magnets, quantum impurity models, and quantum chromodynamics with supplemental four-fermion interactions. This broad spectrum of physical systems provides an ideal setting to explore the proposed fixed-point-annihilation mechanism and its interrelations with other concepts, such as fractionalization. In a sense, this is parallel to mathematics, where different proofs of a single theorem are not merely repetitions of the same task, but reveal distinct facets of the underlying structure. This is because different proofs may employ different proof schemes, may rely on different theorems, or may suggest different generalizations, capturing the essence of Atiyah’s quote above. In a similar way, we aim to understand the proposed fixed-point-annihilation mechanism through its different realizations.

This chapter is structured as follows. First, we motivate this project, see Sec. 3.1, and afterwards propose a novel fixed-point-annihilation mechanism generating generic, unam-

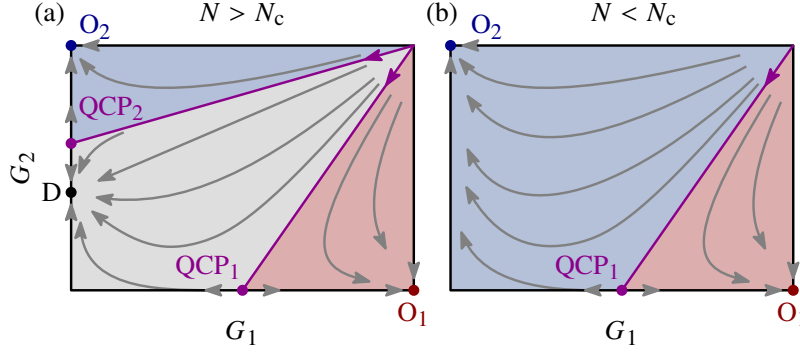
biguously continuous order-to-order phase transitions, see Sec. 3.2. Second, we discuss a concrete realization of the proposed mechanism in Luttinger fermion systems in three spatial dimensions, see Sec. 3.3, with the help of a mean-field analysis, see Sec. 3.4, and a renormalization group analysis, see Sec. 3.5. A comparison with other fixed-point-annihilation scenarios in the literature is given, and unique experimental signatures of the proposed mechanism are derived. Third, we discuss further realizations of the mechanism in kagome quantum magnets, see Sec. 3.6, quantum impurity models, see Sec. 3.7, and quantum chromodynamics, see Sec. 3.8. These additional realizations help develop an intuition for the proposed mechanism and the systems where it may manifest, clarify the role of fractionalization, and provide an opportunity to highlight a recent numerical observation. We conclude in Sec. 3.9.

### 3.1. Motivation

The exploration of quantum phase transitions beyond the Landau-Ginzburg-Wilson paradigm has garnered significant attention in recent years. As discussed in Sec. 1.4, a particular focus has been on continuous quantum phase transitions between phases that break different symmetries [161]. Traditionally, it has been believed that such transitions require fractionalization, where new deconfined quasiparticles with fractional quantum numbers arise and interact via emergent gauge fields. The intense study of deconfined quantum phase transitions in recent years has significantly advanced our understanding of quantum critical phenomena. Notable insights include the discovery of emergent symmetries [232, 246, 268, 371], the identification of spectroscopic signatures of fractionalization [372], the uncovering of unexpected webs of field-theoretical dualities [242], and the observation of anomalous scaling behaviors in entanglement entropy [153, 373]. However, the most recent numerical studies suggest that the deconfined quantum phase transition, at least in its original formulation within  $SU(2)$  spin models [161, 228, 229], is ultimately first-order, characterized by a finite, though large, correlation length at the transition point [153, 233, 268–270], cf. Sec. 1.4.3. The goal of this chapter is to propose a novel mechanism that generates generic, unambiguously continuous order-to-order phase transitions and to exemplify this mechanism through explicit calculations in various systems. Because the proposed mechanism is distinct from the fractionalization mechanism, it uncovers an entirely new class of phase transitions beyond the Landau-Ginzburg-Wilson paradigm, namely, continuous order-to-order phase transitions from fixed-point annihilation, cf. Fig. 1.10.

### 3.2. Fixed-point-annihilation mechanism

In this section, we illustrate a mechanism that provides an alternative pathway to continuous order-to-order quantum phase transitions, one that does not rely on fractionalization. Our proposal involves the collision and annihilation of two renormalization group fixed points: a quantum critical fixed point and an infrared stable fixed point. We argue that



**Figure 3.1.:** Mechanism for continuous order-to-order transition from fixed-point annihilation. (a) Schematic renormalization group flow in the space spanned by two couplings,  $G_1$  and  $G_2$ , for  $N > N_c$ . Arrows indicate flow towards infrared. Beyond an interacting disordered phase (gray) at small couplings, governed by the fixed point D, there are two ordered phases (red and blue) in the strong-coupling regime, governed by the fixed points  $O_1$  and  $O_2$ . The disordered phase is separated from the two ordered phases by continuous phase transition lines (purple), governed by the quantum critical fixed points  $QCP_1$  and  $QCP_2$ . As  $N$  decreases, the fixed points D and  $QCP_2$  approach each other and eventually collide at a critical value  $N_c$ . (b) Same as (a), but for  $N < N_c$ . The fixed-point annihilation at  $N_c$  alters the flow topology, eliminating the disordered phase and leaving behind a single continuous phase transition line (purple) between two phases with distinct orders. Figure reprinted from Ref. [283].

the mechanism is relevant to a broad spectrum of physical systems. In particular, we explicitly demonstrate within a renormalization group calculation that three-dimensional Luttinger fermion systems undergo a continuous quantum phase transition between an antiferromagnetic Weyl semimetal state, marked by broken time-reversal symmetry, and a nematic topological insulator state, characterized by broken lattice rotational symmetry, see Secs. 3.3–3.5. Further possible realizations in other setups are discussed as well, see Secs. 3.6–3.8.

The mechanism is schematically illustrated in Fig. 3.1, which shows renormalization group flow diagrams for two different values of an external control parameter  $N \in \mathbb{R}$ . For  $N$  larger than a critical value  $N_c$ , there is an interacting disordered phase, which is separated from two ordered phases by continuous phase transition lines, see Fig. 3.1(a). As  $N$  decreases, the fixed point associated with the disordered phase (D) and those governing one of the two transition lines ( $QCP_2$ ) approach each other and eventually collide at a critical value  $N_c$ . For  $N$  smaller than  $N_c$ , Fig. 3.1(b), the fixed-point annihilation alters the flow topology by removing the disordered phase, leaving behind a continuous phase transition line between two phases with distinct orders. Note that the  $QCP_1$  fixed point, which governs the resulting continuous order-to-order transition, is entirely decoupled from the fixed-point annihilation occurring between D and  $QCP_2$ . On the left-hand side of  $QCP_1$ , for  $G_1$  below the fixed-point value, the system would naturally flow toward the disordered phase. However, due to the annihilation of the disordered fixed point D with  $QCP_2$ , this phase becomes unstable, giving rise instead to the competing order. Notably, since the proposed mechanism solely relies on the fixed-point annihilation of the fixed points D and  $QCP_2$  and the resulting rearrangement of the flow topology, any quantum critical fixed point not involved in the annihilation qualifies as  $QCP_1$ .

Let us end this section with a few general remarks. First, we have outlined the mechanism using two quantum critical fixed points,  $\text{QCP}_1$  and  $\text{QCP}_2$ , since continuous order-to-order quantum phase transitions are the main focus of this chapter. However, we emphasize that the proposed mechanism could in principle also apply to thermal phase transitions, realized through two classical critical fixed points, say,  $\text{CP}_1$  and  $\text{CP}_2$ . This is different from the traditional fractionalization mechanism, which yields only continuous order-to-order *quantum* phase transitions. Moreover, note that the proposed fixed-point-annihilation mechanism is not only of interest for continuous phase transitions between two long-range-ordered phases, but expected to encompass further exotic phase transitions. This includes transitions between a long-range-ordered phase and a topologically ordered phase, and even transitions between two phases with distinct topological orders. In both cases, the systems must feature a suitable fixed-point annihilation of the fixed points  $\text{D}$  and  $\text{QCP}_2$ . Another exotic scenario appears to arise when the ordered fixed point  $\text{O}_1$ , cf. Fig. 3.1, is replaced by a line of fixed points associated with a quasi-long-range-ordered phase, as discussed in Sec. 1.2.1. The critical endpoint  $\text{QCP}_1$  is expected to govern a transition between a quasi-long-range-ordered phase and a genuine long-range-ordered phase, provided the system features a suitable fixed-point annihilation in the other, symmetry-breaking, sector. Thus, rather than viewing the proposed fixed-point-annihilation mechanism as a specialized mechanism for continuous phase transitions between two long-range-ordered phases, it is helpful to view it as a mechanism for various exotic critical scenarios, enabled simply by a change in the flow topology. Lastly, for a comparison to deconfined pseudocriticality arising from fixed-point annihilation, we refer the reader to Sec. 3.5.1 (last paragraph).

### 3.3. Luttinger semimetals

In this section, we introduce a generalization of the Luttinger fermion model encountered in Chap. 2. This generalized theory is subsequently used to illustrate the proposed fixed-point-annihilation mechanism and is presented here in two equivalent formulations: a purely fermionic formulation, see Sec. 3.3.1, and a partially bosonized formulation, see Sec. 3.3.2.

#### 3.3.1. Fermionic formulation

We begin by reviewing the interacting Luttinger fermion model in three spatial dimensions in the fermionic formulation [328–330, 332]. As elaborated in Chap. 2, this continuum model effectively describes electronic excitations near a quadratic band touching point at the Fermi level, as they occur in pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ , with  $R$  a rare-earth element [303, 311–313, 318]. In the low-temperature limit, the model can be defined by

the Euclidean action  $S = S_0 + S_{\text{int}}$ , where  $S_0$  corresponds to the noninteracting part,

$$S_0 = \int d\tau d^3x \sum_{i=1}^N \psi_i^\dagger \left[ \partial_\tau + \sum_{a=1}^5 (1 + s_a \delta) d_a (-i\nabla) \gamma_a \right] \psi_i, \quad (3.1)$$

which originates in the Luttinger Hamiltonian [333]. As before, cf. Sec. 2.2,  $N$  corresponds to the number of band touching points at the Fermi level, with  $N = 1$  in the case of the pyrochlore iridates,  $\psi_i$  is a four-component Grassmann field,  $s_a := 1$  ( $s_a := -1$ ) for  $a = 1, 2, 3$  ( $a = 4, 5$ ),  $\delta$  parametrizes the cubic anisotropy with  $-1 \leq \delta \leq 1$ , the  $4 \times 4$  matrices  $\gamma_a$  fulfill the Euclidean Clifford algebra,  $\{\gamma_a, \gamma_b\} = 2\delta_{ab}\mathbb{1}$ , and  $d_1(\vec{p}) := \sqrt{3}p_y p_z$ ,  $d_2(\vec{p}) := \sqrt{3}p_x p_z$ ,  $d_3(\vec{p}) := \sqrt{3}p_x p_y$ ,  $d_4(\vec{p}) := \sqrt{3}(p_x^2 - p_y^2)/2$ , and  $d_5(\vec{p}) := (2p_z^2 - p_x^2 - p_y^2)/2$  are proportional to the  $\ell = 2$  real spherical harmonics.

We account for both short-range repulsion and the long-range tail of the Coulomb interaction via [281, 282, 301, 326, 327, 331, 334]

$$S_{\text{int}} = - \int d\tau d^3x \left[ \frac{G_1}{2N} (\psi^\dagger \gamma_{45} \psi)^2 + \frac{G_2}{2N} \sum_{a=1}^5 (\psi^\dagger \gamma_a \psi)^2 + \frac{e^2}{8\pi N} \int d^3y \psi^\dagger(\vec{x}) \psi(\vec{x}) \frac{1}{|\vec{x} - \vec{y}|} \psi^\dagger(\vec{y}) \psi(\vec{y}) \right], \quad (3.2)$$

where we defined  $\gamma_{45} := i\gamma_4\gamma_5$ . For notational simplicity, we abbreviated  $\psi \equiv \psi_i(\tau, \vec{x})$  and  $\psi(\vec{x}) \equiv \psi_i(\tau, \vec{x})$ , and suppressed the summation symbols over  $i = 1, \dots, N$ . The long-range tail parameterized by the charge  $e^2$  must be taken into account because the Coulomb interaction is only marginally screened due to the vanishing density of states at the Fermi level [334]. The coupling  $G_1$  originates from local Hubbard repulsion [335], and is believed to increase with decreasing radius of the rare-earth ion [303]. As shown in Chap. 2, a sizable  $G_1$  generates a transition towards a Weyl semimetal phase characterized by spontaneous time-reversal symmetry breaking and all-in-all-out (AIAO) antiferromagnetic order on the pyrochlore lattice [301, 326, 327, 335]. The coupling  $G_2$  may be assumed to be small on the microscopic level. However, as will be shown below,  $G_2$  is generated by  $G_1$  and the long-range interaction, and we therefore include it from the outset. A sizable  $G_2$  drives a transition towards a nematic phase characterized by spontaneous lattice rotational symmetry breaking and a topological gap in the fermionic spectrum [281, 282, 331]. While previous works have focused exclusively on instabilities toward either AIAO order [301, 326, 327, 335] or nematic order [281, 282, 331], the present study treats both order parameters on equal footing. This proves essential for revealing the direct order-to-order transition.

### 3.3.2. Hubbard-Stratonovich formulation

Having introduced the Luttinger model in the purely fermionic formulation above, we now show how the interactions can be rewritten in terms of Hubbard-Stratonovich fields  $a$ ,  $\phi$ , and  $\varphi$ . The Hubbard-Stratonovich formulation of the model is useful for the mean-field

and renormalization group analyses. In particular, this formulation is crucial in order to explicitly compute order parameters within the renormalization group approach, since it avoids the runaway flows encountered in the purely fermionic formulations [281, 282, 327], see also Sec. 3.5.6.

We introduce the scalar Coulomb field  $a$  via

$$S_a = \int d\tau d^3x \left[ \frac{1}{2}(\nabla a)^2 + \frac{ie}{\sqrt{N}} a \psi_i^\dagger \psi_i \right], \quad (3.3)$$

where the summation over repeated indices  $i = 1, \dots, N$  has been implicitly assumed. This action aligns with Eq. (2.2) upon rescaling the charge  $e \mapsto \sqrt{N}e$ . AIAO fluctuations are parametrized by an Ising order-parameter field  $\phi$ , which couples to the electronic quasiparticles as [301, 326, 327, 335]

$$S_\phi = \int d\tau d^3x \left[ \frac{1}{2} \phi(r_1 - \nabla^2) \phi + \frac{g_1}{\sqrt{N}} \phi \psi_i^\dagger \gamma_{45} \psi_i \right], \quad (3.4)$$

where  $r_1$  represents the tuning parameter for the AIAO transition and  $g_1$  is the AIAO Yukawa coupling. Again, this action coincides with Eq. (2.3) upon rescaling the coupling  $g_1 \mapsto \sqrt{N}g_1$ . Nematic fluctuations are parametrized by a five-component tensor order-parameter field  $\varphi_a$ , which couples to the electronic quasiparticles as [282, 331, 334]

$$S_\varphi = \int d\tau d^3x \left[ \frac{1}{2} \sum_{a=1}^5 \varphi_a (r_2 - c \partial_\tau^2 - \nabla^2) \varphi_a + \frac{g_2}{\sqrt{N}} \sum_{a=1}^5 \varphi_a \psi_i^\dagger \gamma_a \psi_i + \frac{\lambda}{\sqrt{N}} \sum_{a,b,c=1}^5 \text{tr}(\Lambda_a \Lambda_b \Lambda_c) \varphi_a \varphi_b \varphi_c \right], \quad (3.5)$$

where  $r_2$  represents the tuning parameter for the nematic transition,  $g_2$  is the nematic Yukawa coupling, and  $\lambda$  parametrizes the bosonic self-interaction.  $\Lambda_a$  denote the five real Gell-Mann matrices in three spatial dimensions<sup>1</sup>. The parameter  $c$  is power-counting irrelevant and included for regularization purposes. Quartic bosonic self-interactions are also power-counting irrelevant and are therefore set to zero from the beginning.

The fermion-boson model defined by the Euclidean action

$$S_{\text{FB}} = S_0 + S_a + S_\phi + S_\varphi \quad (3.6)$$

is quadratic in the Hubbard-Stratonovich fields  $a$ ,  $\phi$ , and  $\varphi$ . The latter can therefore, in principle, be integrated out. The original microscopic, purely fermionic action  $S = S_0 + S_{\text{int}}$  presented in Sec. 3.3.1 can then be recovered from the above fermion-boson action in the

<sup>1</sup>We follow the notation of Ref. [301]. The five real Gell-Mann matrices and therefore also the  $\ell = 2$  real spherical harmonics used in Ref. [331] correspond to the ones used here by relabelling indices  $(1, 2, 3, 4, 5) \mapsto (4, 3, 2, 1, 5)$ .

limit of infinite order-parameter masses  $r_{1,2} \rightarrow \infty$  and  $g_{1,2}^2 \rightarrow \infty$ , with fixed ratios

$$G_1 = \frac{g_1^2}{r_1} \quad \text{and} \quad G_2 = \frac{g_2^2}{r_2}, \quad (3.7)$$

and with  $c = \lambda = 0$ .

### 3.4. Mean-field analysis

In the large- $N$  limit and for small  $G_1$  and  $G_2$ , the couplings flow to an infrared stable renormalization group fixed point, characterizing an interacting semimetal phase. This phase, which realizes a three-dimensional non-Fermi liquid, is known as the Luttinger-Abrikoso-Beneslavskii (LAB) phase [328–330]. Sizable four-fermion interactions beyond a finite threshold, however, drive instabilities of the LAB state, which can be understood within a mean-field analysis. To this end, we integrate out the fermion fields  $\psi_i^\dagger$  and  $\psi_i$ ,  $i = 1, \dots, N$ , in Eq. (3.6), leading to an effective action for the Hubbard-Stratonovich fields, cf. Sec. 2.3. In the large- $N$  limit, the saddle point of this effective action dominates the partition function. Determining the saddle-point solution ultimately reduces to minimizing the mean-field energy [301, 331, 339],

$$E_{\text{MF}}(\phi, \varphi) = \frac{r_1}{2}\phi^2 + \frac{r_2}{2}\varphi^2 + N \int_0^\Lambda \frac{d^3p}{(2\pi)^3} [\varepsilon_{\phi,\varphi}^+(\vec{p}) + \varepsilon_{\phi,\varphi}^-(\vec{p})] \quad (3.8)$$

with respect to the order-parameter fields  $\phi$  and  $\varphi$ . Here, we used the fact that minimization of the mean-field energy implies the Coulomb field vanishes at the mean-field level, and assumed that the nematic order parameter takes the form  $(\varphi_a) = (0, 0, 0, 0, \varphi)$ . This form implies that the nematic phase is uniaxial, characterized by a single director with a residual discrete rotational symmetry remaining intact in the plane perpendicular to the director, and is believed to realize the smallest mean-field energy among the different possible nematic states [331].  $\Lambda$  is the ultraviolet cutoff. The integrands  $\varepsilon_{\phi,\varphi}^\pm(\vec{p})$  denote the two lower-branch energy eigenvalues of the mean-field Hamiltonian

$$H_{\text{MF}} = \sum_{a=1}^5 (1 + s_a \delta) d_a(\vec{p}) \gamma_a + \frac{g_1}{\sqrt{N}} \phi \gamma_{45} + \frac{g_2}{\sqrt{N}} \varphi \gamma_5. \quad (3.9)$$

They are given as

$$\varepsilon_{\phi,\varphi}^\pm(\vec{p}) = - \left\{ (1 - \delta)^2 p^4 + 4\delta \sum_{a=1}^3 d_a^2(\vec{p}) + \left( \frac{g_1 \phi}{\sqrt{N}} \right)^2 + \left( \frac{g_2 \varphi}{\sqrt{N}} \right)^2 + 2(1 - \delta) d_5 \frac{g_2 \varphi}{\sqrt{N}} \pm 2(1 + \delta) \frac{|g_1 \phi|}{\sqrt{N}} \left[ \sum_{a=1}^3 d_a^2(\vec{p}) \right]^{1/2} \right\}^{1/2}. \quad (3.10)$$

In Eq. (3.8), the first two terms, proportional to  $r_1$  and  $r_2$ , penalize finite order parameters, while the integral can lower the energy by causing partial or full band gap opening.

In order to expand the mean-field energy around small values of the order parameters, we introduce polar coordinates in field space as  $\phi = (\sqrt{N}/g_1)\chi \cos \alpha$  and  $\varphi = (\sqrt{N}/g_2)\chi \sin \alpha$ , with radial part  $\chi > 0$  and angular part  $\alpha \in [0, 2\pi)$ . A finite  $\chi$  signals an instability of the Luttinger semimetal state, while the minimizing angle determines whether AIAO order ( $\alpha \in \{0, \pi\}$ ), nematic order ( $\alpha \in \{\pi/2, 3\pi/2\}$ ), or a coexistence of both orders is realized ( $\alpha \notin \{0, \pi/2, \pi, 3\pi/2\}$ ). Following Refs. [301, 331], it is convenient to add two suitably written zeros to the mean-field energy, corresponding to the constant and quadratic terms in  $E_{\text{MF}}$  with respect to  $\chi$ , namely

$$0 = \int \frac{d\Omega}{(2\pi)^3} \left[ \frac{1}{\tilde{X}^{3/4}} \int_0^{\tilde{X}^{1/4}\Lambda} \frac{dx}{x^{1/2}} \left( x^4 + \frac{c_{\pm}x^2}{2\tilde{X}^{1/2}} + \frac{1}{2} - \frac{c_{\pm}^2}{8\tilde{X}} \right) \chi^{5/2} \right. \\ \left. - \left( \frac{\tilde{X}^{1/2}\Lambda^5}{5} + \frac{c_{\pm}\Lambda^3\chi}{6\tilde{X}^{1/2}} + \frac{\Lambda\chi^2}{2\tilde{X}^{1/2}} - \frac{c_{\pm}^2\Lambda\chi^2}{8\tilde{X}^{3/2}} \right) \right], \quad (3.11)$$

where  $c_{\pm} := 2(1 - \delta)\tilde{d}_5 \sin \alpha \pm 2(1 + \delta)|\cos \alpha| \sqrt{\sum_{a=1}^3 \tilde{d}_a^2}$  with  $\tilde{d}_a \equiv \tilde{d}_a(\theta, \phi) := d_a/p^2$  the real spherical harmonics,  $\int d\Omega$  the integration over the solid angle in momentum space, and we abbreviated  $\tilde{X}(\theta, \phi) := (1 - \delta)^2 + 12\delta \sum_{i < j} \frac{p_i^2 p_j^2}{p^2}$ . Adding those zeros allows us to express the mean-field energy for small  $\chi \ll \Lambda^2$  as

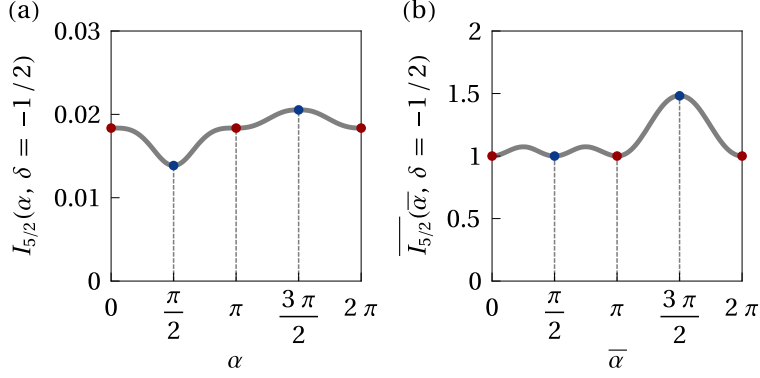
$$\frac{E_{\text{MF}}(\chi, \alpha)}{N\Lambda^5} = \frac{1}{2} \left( \frac{r_1 \cos^2 \alpha}{g_1^2 \Lambda} + \frac{r_2 \sin^2 \alpha}{g_2^2 \Lambda} - I_2(\alpha, \delta) \right) \left( \frac{\chi}{\Lambda^2} \right)^2 \\ + I_{5/2}(\alpha, \delta) \left( \frac{\chi}{\Lambda^2} \right)^{5/2} + \mathcal{O} \left( \left( \frac{\chi}{\Lambda^2} \right)^3 \right), \quad (3.12)$$

where we have subtracted an additive constant, which is irrelevant for the minimization problem, and introduced the dimensionless integrals

$$I_2(\alpha, \delta) := \int \frac{d\Omega}{(2\pi)^3} \left[ -\frac{c_+^2 + c_-^2}{4\tilde{X}^{3/2}} + \frac{2}{\tilde{X}^{1/2}} \right] \quad (3.13)$$

$$= \frac{5f_1(\delta) - 3(1 + \delta)f_{2t}(\delta)}{5\pi^2} \cos^2 \alpha \\ + \frac{5f_1(\delta) - (1 - \delta)f_{2e}(\delta)}{5\pi^2} \sin^2 \alpha, \quad (3.14)$$

$$I_{5/2}(\alpha, \delta) := \int_0^\infty dx \int \frac{d\Omega}{(2\pi)^3} \frac{1}{\tilde{X}^{3/4}} \left[ 2x^4 + \frac{1}{2} \frac{c_+ + c_-}{\tilde{X}^{1/2}} x^2 \right. \\ \left. - \frac{1}{8} \left( \frac{c_+}{\tilde{X}^{1/2}} \right)^2 - \frac{1}{8} \left( \frac{c_-}{\tilde{X}^{1/2}} \right)^2 + 1 \right. \\ \left. - x^2 \sqrt{x^4 + \frac{c_+}{\tilde{X}^{1/2}} x^2 + 1} - x^2 \sqrt{x^4 + \frac{c_-}{\tilde{X}^{1/2}} x^2 + 1} \right]. \quad (3.15)$$

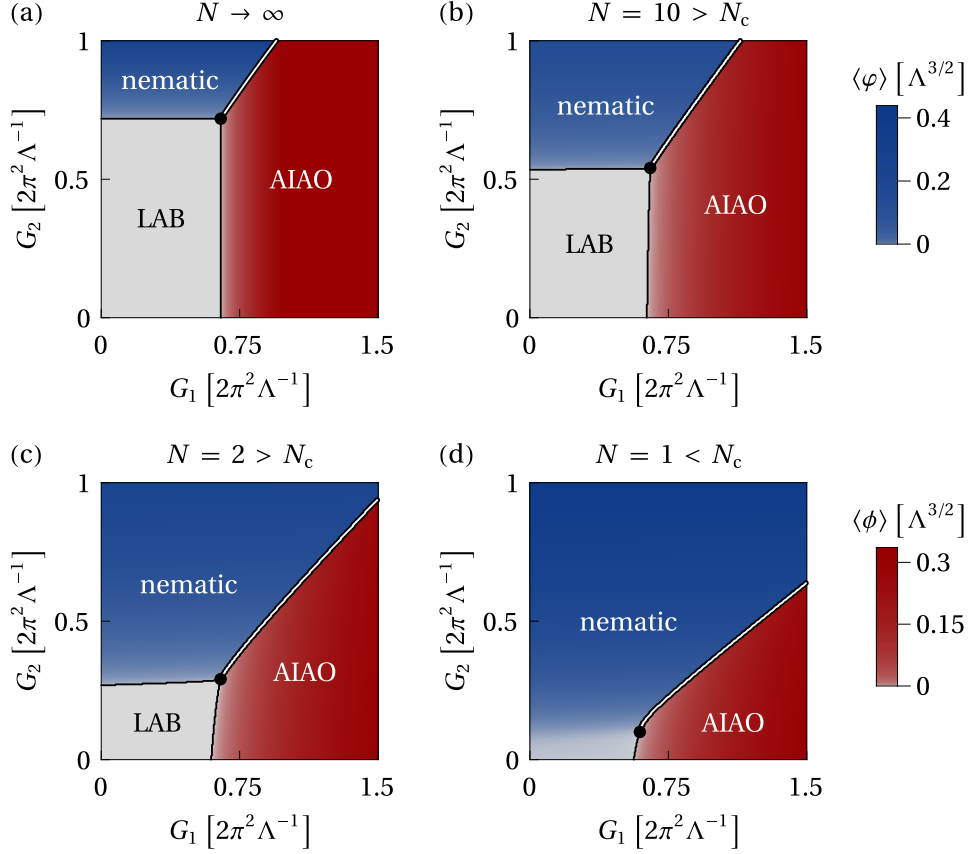


**Figure 3.2.:** (a) Coefficient  $I_{5/2}$  of the nonanalytic term  $\propto \chi^{5/2}$  in the mean-field energy, for fixed anisotropy parameter  $\delta = -1/2$ , as a function of polar angle  $\alpha$  in  $(\phi, \varphi)$  field space. Here,  $\alpha \in \{0, \pi\}$  and  $\alpha \in \{\pi/2, 3\pi/2\}$  correspond to the AIAO and nematic sectors, respectively.  $I_{5/2}$  is positive for all  $\alpha \in [0, 2\pi)$ , justifying the truncation of the mean-field energy after the nonanalytic term. (b) Rescaled coefficient  $\overline{I}_{5/2}$  as a function of rescaled polar angle  $\overline{\alpha}$ , for fixed  $\delta = -1/2$ , unveiling three global minima at  $\overline{\alpha} = 0, \pi$  (AIAO sector) and  $\overline{\alpha} = \pi/2$  (nematic sector). This excludes the possibility of coexisting AIAO and nematic order. Figure reprinted from Ref. [283].

The integral  $I_2(\alpha, \delta)$  can be computed analytically and leads to anisotropy-dependent renormalizations of the tuning parameters  $r_1$  and  $r_2$ . The involved dimensionless functions  $f_1$ ,  $f_{2e}$ , and  $f_{2t}$  stem from solid-angle integration and are defined in App. A. Notably, the coefficient of the quadratic term in  $\chi$  in Eq. (3.12) takes the form  $r'_1 \cos^2 \alpha + r'_2 \sin^2 \alpha$  with dimensionless renormalized tuning parameters  $r'_{1,2}$ . Consequently, the coefficient is minimized for  $\alpha = 0, \pi$  if  $r'_1 < r'_2$  and for  $\alpha = \pi/2, 3\pi/2$  if  $r'_1 > r'_2$ . The nonanalytic term  $I_{5/2}(\alpha, \delta) \chi^{5/2}$  arises from the presence of gapless fermionic degrees of freedom, cf. Sec. 1.3. For general  $(\alpha, \delta)$ , it has to be calculated numerically. Figure 3.2(a) shows  $I_{5/2}$  as a function of  $\alpha$  for a representative fixed value of  $\delta = -1/2$ . Importantly,  $I_{5/2}(\alpha, \delta = -1/2) > 0$  for all values of  $\alpha$ . This implies a continuous transition between the disordered Luttinger semimetal and the long-range-ordered phases, such that the higher-order terms  $\mathcal{O}(\chi^3)$  in Eq. (3.12) can be neglected near the transition. In order to determine the nature of the transition between the two ordered phases, it is useful to rescale the fields  $\phi$  and  $\varphi$  in a way that the coefficient of the nonanalytic term in the AIAO sector is the same as those in the nematic sector. This can be achieved by introducing the rescaled polar angle  $\overline{\alpha}$  using the field redefinitions  $\phi = [\sqrt{N}/(a_1 g_1)] \chi \cos \overline{\alpha}$  and  $\varphi = [\sqrt{N}/(a_2 g_2)] \chi \sin \overline{\alpha}$  with  $a_1 := [I_{5/2}(\alpha = 0)]^{2/5}$  and  $a_2 := [I_{5/2}(\alpha = \pi/2)]^{2/5}$ . Note that  $\alpha = \overline{\alpha}$  for  $\alpha \in \{0, \pi/2, \pi, 3\pi/2\}$ . On the level of the quadratic term  $\propto \chi^2$  in the mean-field energy, the rescaling can be absorbed in a redefinition of the tuning parameters  $r_1$  and  $r_2$ . The rescaled coefficient of the nonanalytic term  $\propto \chi^{5/2}$  now takes the form

$$\overline{I}_{5/2}(\overline{\alpha}, \delta) := \left( \frac{\cos^2 \overline{\alpha}}{a_1^2} + \frac{\sin^2 \overline{\alpha}}{a_2^2} \right)^{5/4} I_{5/2}(\alpha(\overline{\alpha}), \delta). \quad (3.16)$$

Figure 3.2(b) shows  $\overline{I}_{5/2}$  as a function of  $\overline{\alpha}$  for a representative fixed value of  $\delta = -1/2$ . The rescaled coefficient hosts three global minima, lying at angles  $\overline{\alpha} = 0, \pi$  (AIAO sector) as well as  $\overline{\alpha} = \pi/2$  (nematic sector). This implies that a coexistence phase (corresponding to  $\alpha \notin \{0, \pi/2, \pi, 3\pi/2\}$ ) is not realized for all values of  $r'_1$  and  $r'_2$ . As a consequence,



**Figure 3.3.:** (a) Phase diagram of Luttinger model in the low-temperature limit as a function of short-range couplings  $G_1$  and  $G_2$  for fixed representative values of the charge  $e^2 = 3\pi^2\Lambda/2$  and the anisotropy parameter  $\delta = -1/2$  from mean-field theory, which becomes exact in the limit  $N \rightarrow \infty$ . The Luttinger-Abrikosov-Beneslavskii (LAB) phase at small couplings realizes a three-dimensional non-Fermi liquid (gray). For sufficiently large  $G_1$ , a Weyl semimetal emerges, characterized by AIAO antiferromagnetic order on the pyrochlore lattice. For sizable  $G_2$ , a nonmagnetic nematic topological insulator is stabilized. Color scale indicates magnitudes of AIAO order parameter  $\langle \phi \rangle \propto \langle \psi^\dagger \gamma_{45} \psi \rangle$  (red) and nematic order parameter  $\langle \varphi \rangle \propto \langle \psi^\dagger \gamma_5 \psi \rangle$  (blue). Single (double) lines indicate continuous (discontinuous) phase transitions. (b) Same as (a), but for  $N = 10$  from renormalization group analysis, qualitatively agreeing with the mean-field result. (c) Same as (b), but for  $N = 2$ . The LAB phase shrinks in favor of the nematic phase as the LAB fixed point and the quantum critical fixed point associated with the nematic instability approach each other. (d) Same as (b), but for  $N = 1$ , corresponding to the case relevant for the pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ . The LAB phase is removed as a consequence of the fixed-point annihilation, leaving behind a continuous phase transition line between the AIAO antiferromagnetic Weyl semimetal and the nematic topological insulator for small  $G_2$ . Figure reprinted from Ref. [283].

the mean-field theory predicts a direct discontinuous transition between the AIAO and nematic orders.

This analytical result is consistent with the numerical minimization of the mean-field energy, Eq. (3.12), with respect to the mean-field parameters  $\chi$  and  $\alpha$  for given  $G_1 = g_1^2/r_1$ ,  $G_2 = g_2^2/r_2$ , and  $\delta$ . This is illustrated in Fig. 3.3(a), which shows the resulting mean-field phase diagram as a function of  $G_1$  and  $G_2$  for a fixed representative value of  $\delta$ . Beyond the LAB phase at small couplings, we identify two ordered phases: For sufficiently large  $G_1$ , a Weyl semimetal phase emerges, characterized by the Ising order parameter  $\langle \phi \rangle \propto \langle \psi^\dagger \gamma_{45} \psi \rangle$ , which signals spontaneous time-reversal symmetry breaking. On the pyrochlore

lattice, this phase exhibits AIAO antiferromagnetic order [301, 326, 327], cf. Chap. 2. In contrast, for sizable  $G_2$ , we find a nonmagnetic phase that breaks lattice rotational symmetry. This phase is characterized by the nematic order parameter  $\langle \varphi \rangle \propto \langle \psi^\dagger \gamma_5 \psi \rangle$ , constructed from one of the five irreducible components of traceless symmetric tensors in three dimensions [331]. It corresponds to a nematic topological insulator [281, 282]. While the transitions from the LAB phase to the ordered phases are continuous, the direct transition between the two ordered phases is first order, consistent with the Landau-Ginzburg-Wilson paradigm.

## 3.5. Renormalization group analysis

Having discussed the mean-field behavior in the previous section, we now turn to the regime of strong order-parameter fluctuations, employing a renormalization group approach. To keep the presentation as nontechnical as possible, we begin by summarizing the main results of this analysis in Sec. 3.5.1. This includes the illustration of the proposed fixed-point-annihilation mechanism in the form of a discussion of its universal properties, the identification of distinctive experimental signatures, and a comparison with deconfined pseudocriticality. Subsequently, we provide further details for the interested reader. In particular, we present the derivation of the flow equations in Sec. 3.5.2, discuss their fixed-point structure in Sec. 3.5.3, explain how the phase diagrams are obtained by integrating the flow equations in Sec. 3.5.4, investigate higher-loop corrections in Sec. 3.5.5, and discuss the renormalization group flow in the limit of infinite order-parameter masses in Sec. 3.5.6.

### 3.5.1. Essentials

#### Universal aspects

At finite  $N$ , fluctuation effects can become significant. To account for these effects, we use a renormalization group analysis in the dynamical bosonization framework [374–376]. This approach applies a Hubbard-Stratonovich transformation after each renormalization group step, effectively capturing potential nonperturbative effects from the renormalization-group-induced generation of four-fermions couplings while avoiding the unphysical singularities associated with their runaway flow. The method has been successfully applied to Luttinger semimetals [282], demonstrating the existence of a critical value  $N_c$  of quadratic band touching points at the Fermi level, consistent with earlier analyses [281, 334]. For  $N \searrow N_c$ , the infrared stable fixed point corresponding to the LAB phase collides and annihilates with the quantum critical fixed point that governs the transition between the LAB phase and the nematic topological insulator. All the different approaches predict a value of  $N_c$  around 2, and therewith above the physical case  $N = 1$  relevant for the pyrochlore iridates.

Crucially, the dynamically bosonized renormalization group approach allows us to compute the order parameters as functions of the microscopic couplings by integrating out the renormalization group flow. This enables us to determine the nature of the phase transitions explicitly. Details of the method are provided in Sec. 3.5.2 (derivation of flow equations), Sec. 3.5.3 (corresponding fixed-point structure), and Sec. 3.5.4 (integration of flow equations). For large  $N$ , we recover the infrared stable fixed point that governs the LAB phase. Additionally, we locate two other interacting fixed points, each with one renormalization-group-relevant direction and otherwise only renormalization-group-irrelevant directions. By integrating out the renormalization group flow, we identify these fixed points as the quantum critical points for the transitions from the LAB phase towards the AIAO antiferromagnetic Weyl semimetal and the nematic topological insulator, respectively. The resulting quantum phase diagram for a representative value of  $N = 10$  is shown in Fig. 3.3(b) and qualitatively<sup>2</sup> aligns with the mean-field result in Fig. 3.3(a). As  $N$  decreases, the LAB fixed point and the quantum critical fixed point associated with the nematic instability approach each other. This implies that the LAB phase shrinks in the phase diagram in favor of the nematic phase, as exemplified for  $N = 2$  in Fig. 3.3(c). Upon further decreasing  $N$ , the two fixed points eventually collide at a critical value  $N_c = 1.856$ , in agreement with Ref. [282]. The fixed-point annihilation for  $N < N_c$  alters the flow topology by removing the LAB phase. We emphasize that no additional fixed points are involved in the collision; in particular, the quantum critical fixed point associated with the AIAO instability remains within the physical space of real couplings for all  $N \geq 1$ . The resulting quantum phase diagram for the physical case of  $N = 1$ , relevant for the pyrochlore iridates, is shown in Fig. 3.3(d). Importantly, there is now a direct transition between the AIAO Weyl semimetal and the nematic topological insulator upon tuning a single parameter,  $G_1$ . Although the two ordered states break different symmetries, and the Landau-Ginzburg-Wilson paradigm would predict a discontinuous transition, the presence of the quantum critical fixed point associated with the AIAO instability leads to a continuous transition without fine tuning, provided  $G_2$  is not too large.

To illustrate this fact, we show in Fig. 3.4(a) the nematic order parameter  $\langle \varphi \rangle \propto \langle \psi^\dagger \gamma_5 \psi \rangle$  and the AIAO antiferromagnetic order parameter  $\langle \phi \rangle \propto \langle \psi^\dagger \gamma_{45} \psi \rangle$  for  $N = 1$  as functions of the ultraviolet value of  $G_1$  while keeping the ultraviolet values of  $G_2$ ,  $e^2$ , and  $\delta$  fixed. Details on obtaining the order parameters by integrating out the flow equations are provided in Sec. 3.5.4. The two order parameters vanish continuously upon approaching the quantum critical point at  $G_{1c}$  from above and below, respectively. The same behavior is observed for other values of  $G_2$ ,  $e^2$ , and  $\delta$ , provided  $G_2$  is not too large. There is a direct continuous order-to-order quantum phase transition without an intermediate mixed phase and without additional fine tuning. The corresponding quantum critical behav-

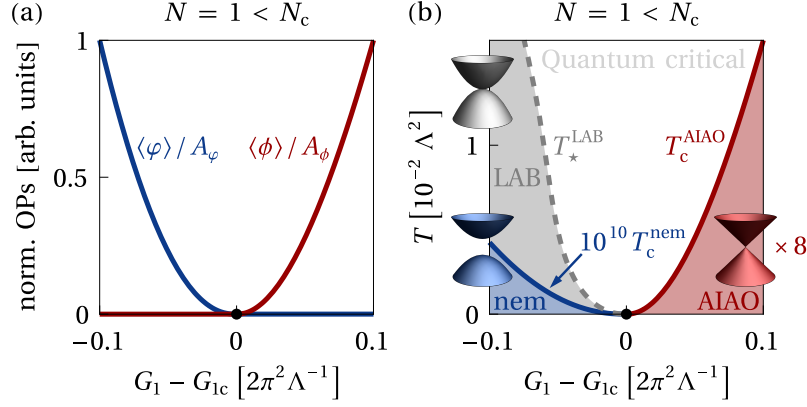
<sup>2</sup>The deviation of the horizontal phase boundaries in Fig. 3.3(a) and (b) arises mainly from slightly different computation schemes (rather than resulting from slight differences in  $1/N$ ). In the mean-field analysis, the spatial anisotropy  $\delta$  is treated exactly, cf. Sec. 3.4. In contrast, the renormalization group analysis presented here treats the spatial anisotropy  $\delta$  only perturbatively in the nematic sector, cf. Sec. 3.5.2. This explains why the slight mismatch occurs between the nematic phase boundaries at  $N \rightarrow \infty$  and  $N = 10$ , but not between the AIAO phase boundaries. We emphasize that the universal properties are unaffected by this approximation in the nematic sector.

ior is governed by the renormalization group flow near the fixed point associated with the AIAO instability. In the present model, this fixed point is situated at extreme cubic anisotropy [326], allowing for the exact computation of all critical exponents [301], as detailed in Sec. 3.5.5. We find that the anomalous dimensions for both the AIAO and nematic orders, respectively are  $\eta_\phi = \eta_\varphi = 1$ , the correlation-length exponents  $\nu_\phi = \nu_\varphi = 1$ , and the dynamical exponent  $z = 2$ . Assuming hyperscaling holds, we find that the two order parameters scale as  $\langle\phi\rangle \propto (G_1 - G_{1c})^{\beta_\phi}$  and  $\langle\varphi\rangle \propto (G_{1c} - G_1)^{\beta_\varphi}$  as  $G_1$  approaches  $G_{1c}$  from above and below, respectively, with the same exponent  $\beta_\phi = \beta_\varphi = 2$ . In Sec. 3.5.4, we replot the data from Fig. 3.4 using a double-logarithmic scale, confirming this predicted scaling of the order parameters. Lastly, we refer the reader to Sec. 3.5.6 for a discussion of the renormalization group flow in the limit of infinite order-parameter masses. The point-like limit allows for several (successful) consistency checks with previous results in the literature and helps build intuition about how the proposed fixed-point-annihilation mechanism manifests itself in fermionic renormalization group flows, thereby preparing us for the examples discussed in Sec. 3.6 and Sec. 3.8.

### Asymmetry in energy scales

Although the two order parameters scale with the same exponents, their corresponding prefactors differ significantly. Since the ratio of these prefactors depends on microscopic parameters and is as such nonuniversal, it cannot be quantitatively predicted within our continuum model. However, at equal distances from criticality, we qualitatively find that the nematic order parameter for  $G_1 < G_{1c}$  is typically several order of magnitude smaller than the AIAO order parameter for  $G_1 > G_{1c}$ . This phenomenon arises as a direct consequence of the fixed-point annihilation occurring at  $N_c = 1.856$ , which is only slightly above the physical value of  $N = 1$ . For  $N$  just below  $N_c$ , the complex LAB fixed point has a small imaginary part. As a result, the renormalization group flow near the location in parameter space where the fixed-point annihilation has occurred becomes logarithmically slow. This slow flow leads to an exponential suppression of the effective energy scale associated with the nematic order, a phenomenon known as walking behavior in the field-theory literature [272, 273]. Walking behavior is observed in fixed-point annihilation scenarios across various systems, including Abelian Higgs models [204, 215] (see Sec. 1.3.2), Wess-Zumino-Witten models [271, 274, 275] (see Sec. 1.4.3), Potts models [277–280], quantum electrodynamics in three space-time dimensions [217, 284–286] (see Sec. 3.6), quantum impurity models [287–290] (see Sec. 3.7), and quantum chromodynamics in four space-time dimensions [276, 291–294] (see Sec. 3.8).

The asymmetry in the effective energy scales on the two different sides of the continuous order-to-order transition is illustrated in the finite-temperature phase diagram shown in Fig. 3.4(b). Details on obtaining the phase diagram are provided in Sec. 3.5.4. On the nematic side of the transition, the critical temperature is significantly suppressed by several orders of magnitude in comparison with the AIAO side. This suppression of the nematic order creates a broad intermediate regime at finite temperature, separated from the



**Figure 3.4.:** (a) Normalized order parameters (OPs)  $\langle\varphi\rangle/A_\varphi$  and  $\langle\phi\rangle/A_\phi$  for nematic (blue) and AIAO (red) order, respectively, for  $N = 1$  as functions of the ultraviolet value of the short-range coupling  $G_1$  from renormalization group analysis for representative fixed ultraviolet values of the short-range coupling  $G_2 = 0$ , charge  $e^2 = 3\pi^2\Lambda/2$ , and anisotropy parameter  $\delta = -1/2$ . There is a direct continuous transition at  $G_{1c} = 0.565 \cdot 2\pi^2/\Lambda$  between the nematic topological insulator for  $G_1 < G_{1c}$  and the AIAO Weyl semimetal for  $G_1 > G_{1c}$ . The normalization factors were chosen such that  $\langle\phi\rangle = A_\phi$  and  $\langle\varphi\rangle = A_\varphi$  for  $G_1 - G_{1c} = \pm 0.1 \cdot 2\pi^2/\Lambda$ , respectively. The order parameters scale as  $\langle\phi\rangle \propto (G_1 - G_{1c})^{\beta_\phi}$  and  $\langle\varphi\rangle \propto (G_{1c} - G_1)^{\beta_\varphi}$  for  $G_1$  approaching  $G_{1c}$  from above and below, respectively, with the same exponent  $\beta_\phi = \beta_\varphi = 2$ , but with significantly different prefactors. (b) Finite-temperature phase diagram for  $N = 1$  as a function of  $G_1$  from renormalization group analysis, with ultraviolet values of  $G_2$ ,  $e^2$ , and  $\delta$  as in (a). Solid (dashed) lines denote phase transitions (crossovers). The quantum critical point between the nematic topological insulator (blue) and the AIAO Weyl semimetal (red) is marked as black dot. The slow renormalization group flow resulting from fixed-point annihilation causes a striking anisotropy between the two sides of the transitions, with the critical temperature significantly suppressed on the nematic side. This suppression of the nematic order leaves an intermediate region at finite temperature dominated by LAB-like behavior (gray). Figure reprinted from Ref. [283].

nonuniversal high-energy regime by a crossover scale  $T_\star^{\text{LAB}}$ . In this regime, the renormalization group flow is “stuck” in the vicinity of the, now imaginary, LAB fixed point [281, 282], and thermodynamic and transport observables exhibit power laws with nontrivial exponents, characteristic of a non-Fermi liquid [330]. While the asymmetry in the effective energy scales on either side of the transition decreases with increasing  $G_2$ , we find it to remain significant for all parameter sets that we have studied. This indicates that it can serve as a distinctive signature for experimentally identifying the proposed mechanism for continuous order-to-order transitions from fixed-point annihilation.

### Comparison with deconfined pseudocriticality

Let us end this first subsection with a brief comparison of the present fixed-point annihilation in Luttinger semimetals and the fixed-point annihilation occurring in the  $N$ -component Abelian Higgs model in three space-time dimensions [204, 205, 215, 377], cf. Sec. 1.3.2. In both cases, fixed-point annihilations occur by lowering the number of degrees of freedom below a certain critical number  $N_c$ . However, as established above, Luttinger semimetals realize a collision and annihilation of a critical fixed point with a *stable* fixed point, while the Abelian Higgs model realizes a collision and annihilation of a critical fixed

point with a *bicritical* fixed point. Without additional fine tuning, a continuous order-to-order transition in a system described by the  $N$ -component Abelian Higgs model for  $N < N_c$  is therefore not possible. This is consistent with numerical results on  $SU(N)$  spin models on two-dimensional lattices, which are believed to be described by the  $N$ -component Abelian Higgs model [153, 236], cf. Sec. 1.4. For  $N = 2$ , the quantum phase transition between a Néel antiferromagnet and a valence bond solid appears weakly first order, which has been associated with the disappearance of the quantum critical fixed point in the Abelian Higgs model due to the fixed-point collision [153, 161, 246]. Thus, ironically, it is the very fixed-point collision mechanism that is responsible for the continuous order-to-order transition in the Luttinger semimetals (where it destabilizes the LAB phase) that leads to a weakly-first-order transition in the  $SU(N)$  spin models (where it destabilizes the required quantum critical fixed point). Similar comparisons to fixed-point annihilations in  $Q$ -state Potts models [277–280] and tensor  $O(N)$  models [378–381] can be made.

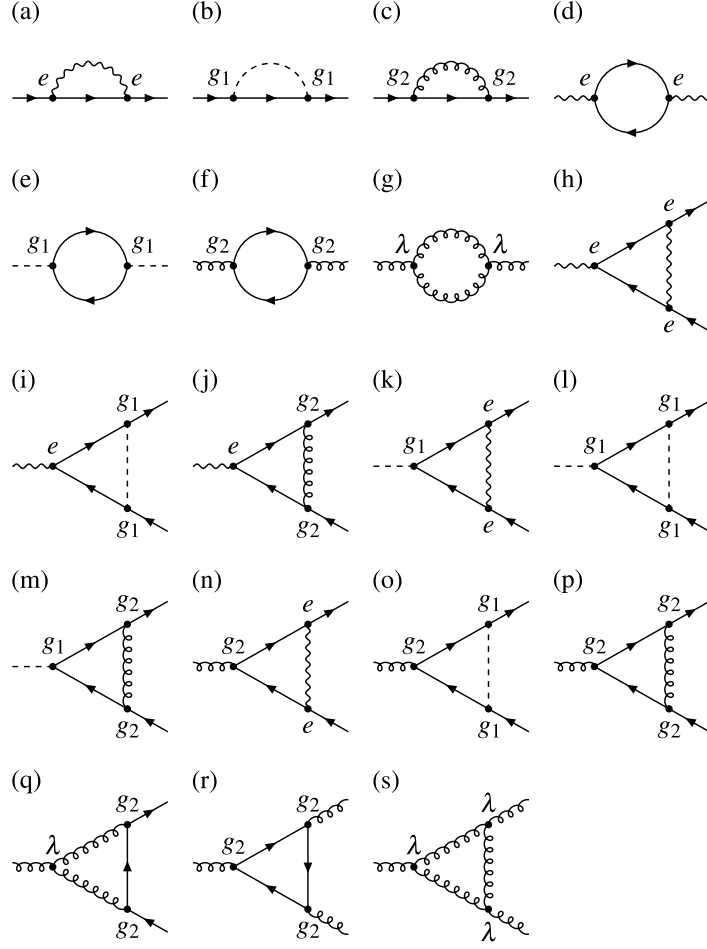
### 3.5.2. Derivation of flow equations

In this subsection, we provide details on the derivation of the flow equations. We start with the flow equations obtained from a standard  $4 - \epsilon$  expansion at the one-loop order and subsequently extend them to the nonperturbative regime,  $\epsilon = 1$ , by employing a dynamical bosonization scheme.

#### 4 - $\epsilon$ expansion

To begin with, note that the fermion-boson model defined in Eq. (3.6) features a unique upper critical dimension. This is because the effective charge  $e$ , the AIAO Yukawa coupling  $g_1$ , the nematic Yukawa coupling  $g_2$ , and the nematic self-interaction  $\lambda$  become simultaneously marginal in  $d = 4$  spatial dimensions. The presence of a unique upper critical dimension enables a controlled  $4 - \epsilon$  expansion. To generalize our theory to non-integer spatial dimensions  $2 < d < 4$ , we maintain the general dimensional scaling of the couplings, but perform angular integrations directly in the physical dimension  $d = 3$  [340]. This approach is considered the most suitable for models of this type [331]. By integrating out modes with momenta  $q$  in the thin shell  $\Lambda/b < q < \Lambda$ , where  $\Lambda$  is the ultraviolet cutoff, and arbitrary frequencies, the renormalized action takes the form

$$\begin{aligned}
S_{\text{FB}}^{\leq} = & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} \left\{ \sum_{i=1}^N \psi_i^\dagger \left[ b^{\eta_1} i\omega + b^{\eta_\psi} \sum_{a=1}^5 d_a(\vec{q}) \gamma_a + (\delta + \Delta\delta) \sum_{a=1}^5 s_a d_a(\vec{q}) \gamma_a \right] \psi_i \right. \\
& + \frac{1}{2} a b^{\eta_a} q^2 a + \frac{1}{2} \phi \left[ b^{\eta_\phi} q^2 + (r_1 + \Delta r_1) \right] \phi \\
& \left. + \frac{1}{2} \sum_{a=1}^5 \varphi_a \left[ b^{\eta_\varphi} q^2 + (c + \Delta c) \omega^2 + (r_2 + \Delta r_2) \right] \varphi_a \right\}
\end{aligned}$$



**Figure 3.5.:** Feynman diagrams at one-loop order contributing to (a)–(c) the fermion anomalous dimensions  $\eta_1$ ,  $\eta_\psi$  and the anisotropy parameter renormalization  $\Delta\delta$ , (d) the Coulomb anomalous dimension  $\eta_a$ , (e) the AIAO order-parameter anomalous dimension  $\eta_\phi$  and the AIAO tuning parameter renormalization  $\Delta r_1$ , (f,g) the nematic order-parameter anomalous dimension  $\eta_\varphi$ , the parameter renormalization  $\Delta c$ , and the nematic tuning parameter renormalization  $\Delta r_2$ , (h)–(j) the charge renormalizations  $\Delta e$ , (k)–(m) the AIAO Yukawa vertex renormalizations  $\Delta g_1$ , (n)–(p) the nematic Yukawa vertex renormalizations  $\Delta g_2$ , (q)–(s) the self-interaction renormalizations  $\Delta\lambda$ . The contributions to the flow of  $e^2$  from (a) and (h), from (b) and (i), and from (c) and (j), respectively, cancel as a consequence of a Ward identity. Figure reprinted from Ref. [283].

$$\begin{aligned}
 & + \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{(2\pi)^2} \int_0^{\Lambda/b} \frac{d^d q_1 d^d q_2}{(2\pi)^{2d}} \left[ i \frac{e + \Delta e}{\sqrt{N}} a \sum_{i=1}^N \psi_i^\dagger \psi_i + \frac{g_1 + \Delta g_1}{\sqrt{N}} \phi \sum_{i=1}^N \psi_i^\dagger \gamma_{45} \psi_i \right. \\
 & \left. + \frac{g_2 + \Delta g_2}{\sqrt{N}} \sum_{a=1}^5 \varphi_a \sum_{i=1}^N \psi_i^\dagger \gamma_a \psi_i + \frac{\lambda + \Delta\lambda}{\sqrt{N}} \sum_{a,b,c=1}^5 \text{tr}(\Lambda_a \Lambda_b \Lambda_c) \varphi_a \varphi_b \varphi_c \right]. \quad (3.17)
 \end{aligned}$$

We rescale all frequencies as  $b^z \omega \mapsto \omega$ , with dynamical exponent  $z = 2 + \eta_1 - \eta_\psi$ , and all momenta as  $b \vec{q} \mapsto \vec{q}$ . Further, we renormalize the fields according to  $b^{-(2+d+z-\eta_\psi)/2} \psi \mapsto \psi$ ,  $b^{-(2+d+z-\eta_a)/2} a \mapsto a$ ,  $b^{-(2+d+z-\eta_\phi)/2} \phi \mapsto \phi$ , and  $b^{-(2+d+z-\eta_\varphi)/2} \varphi_a \mapsto \varphi_a$ . The effective action then becomes of the same form as the original action, but with renormalized parameters  $\delta$ ,  $e$ ,  $g_1$ ,  $r_1$ ,  $g_2$ ,  $\lambda$ ,  $c$ , and  $r_2$ . We take into account all one-loop self-energy and vertex diagrams depicted in Fig. 3.5. The resulting flow equations read

$$\begin{aligned}
\frac{d\delta}{d \ln b} &= -\eta_\psi \delta + \frac{2}{15N}(1-\delta^2)[(1+\delta)f_{1t} - (1-\delta)f_{1e}]e^2 \\
&\quad - \frac{2}{15N}(1-\delta^2)[(1+\delta)f_{1t} + (1-\delta)f_{1e}]\frac{g_1^2}{(1+r_1)^3} \\
&\quad + \frac{2}{5N}[(1+\delta)f_{1t} - (1-\delta)f_{1e}]\frac{g_2^2}{(1+r_2)^3},
\end{aligned} \tag{3.18}$$

$$\frac{de^2}{d \ln b} = \left( z + 2 - d - \eta_a + \frac{2\delta}{1-\delta^2} \frac{d\delta}{d \ln b} \right) e^2, \tag{3.19}$$

$$\begin{aligned}
\frac{dg_1^2}{d \ln b} &= \left( 6 - d - z - \eta_\phi - 2\eta_\psi + \frac{2\delta}{1-\delta^2} \frac{d\delta}{d \ln b} \right) g_1^2 \\
&\quad + \frac{2}{5N}(1-\delta)(1-\delta^2)f_{2e}g_1^2e^2 \\
&\quad - \frac{2}{5N}(1-\delta)(1-\delta^2)f_{2e}\frac{g_1^4}{1+r_1} \\
&\quad - \frac{2}{5N}(1-\delta)f_{2e}\frac{g_1^2g_2^2}{1+r_2},
\end{aligned} \tag{3.20}$$

$$\frac{dr_1}{d \ln b} = (2 - \eta_\phi)r_1 - \frac{4}{5}(1-\delta)(1-\delta^2)f_{2e}g_1^2, \tag{3.21}$$

$$\begin{aligned}
\frac{dg_2^2}{d \ln b} &= (6 - d - z^{(0)} - \eta_\varphi^{(0)} - 2\eta_\psi^{(0)})g_2^2 + \frac{12}{5N}\frac{g_2^4}{1+r_2} \\
&\quad + \frac{4}{5N}(1-\delta^2)g_2^2e^2 - \frac{4c_1}{5N}(1-\delta^2)\frac{g_1^2g_2^2}{1+r_1},
\end{aligned} \tag{3.22}$$

$$\frac{d\lambda}{d \ln b} = \frac{1}{2}(6 - d - z^{(0)} - 3\eta_\varphi^{(0)})\lambda - \frac{27}{4N}\frac{\lambda^3}{\sqrt{c}(1+r_2)^{5/2}} - \frac{\sqrt{3}}{35}g_2^3, \tag{3.23}$$

$$\frac{dc}{d \ln b} = (2 - 2z^{(0)} - \eta_\varphi^{(0)})c + \frac{2}{5}g_2^2 + \frac{21}{4N}\frac{\sqrt{c}\lambda^2}{(1+r_2)^{5/2}}, \tag{3.24}$$

$$\frac{dr_2}{d \ln b} = (2 - \eta_\varphi^{(0)})r_2 - \frac{8}{5}g_2^2 - \frac{21}{N}\frac{\lambda^2}{\sqrt{c}(1+r_2)^{3/2}}, \tag{3.25}$$

with the anomalous dimensions given by

$$\begin{aligned}
\eta_\psi &= \frac{2}{15N}(1-\delta^2)[(1+\delta)f_{1t} + (1-\delta)f_{1e}]e^2 \\
&\quad - \frac{2}{15N}(1-\delta^2)[(1+\delta)f_{1t} - (1-\delta)f_{1e}]\frac{g_1^2}{(1+r_1)^3} \\
&\quad + \frac{2}{5N}[(1+\delta)f_{1t} + (1-\delta)f_{1e}]\frac{g_2^2}{(1+r_2)^3},
\end{aligned} \tag{3.26}$$

$$\eta_\psi^{(0)} = \frac{4}{15N}(1-\delta^2)e^2 + \frac{4}{5N}\frac{g_2^2}{(1+r_2)^3}, \tag{3.27}$$

$$\eta_a = e^2 f_{e^2}, \tag{3.28}$$

$$\eta_\phi = g_1^2 f_{g^2}, \tag{3.29}$$

$$\eta_\varphi^{(0)} = \frac{44}{35}g_2^2 + \frac{21}{4N}\frac{\lambda^2}{\sqrt{c}(1+r_2)^{5/2}}, \tag{3.30}$$

$$\eta_1 = 0, \tag{3.31}$$

and the dynamical exponent takes the form

$$z = 2 - \eta_\psi, \quad z^{(0)} = 2 - \eta_\psi^{(0)}. \quad (3.32)$$

In the above flow equations and anomalous dimensions, we have rescaled the couplings as  $(e^2, g_1^2)\Lambda^{-\epsilon}/(2\pi^2) \mapsto (1 - \delta^2)(e^2, g_1^2)$ , which turned out convenient to assess the properties of the AIAO fixed point [301]. We have further introduced the dimensionless couplings  $(g_2^2, \lambda^2)\Lambda^{-\epsilon}/(2\pi^2) \mapsto (g_2^2, \lambda^2)$ , the dimensionless tuning parameters  $(r_1, r_2)\Lambda^{-2} \mapsto (r_1, r_2)$ , and the dimensionless parameter  $c\Lambda^2 \mapsto c$ . The dimensionless functions  $f_i(\delta)$  are defined in App. A. Note that the self-energy contributions to the flow of  $e^2$ , depicted in Figs. 3.5(a)–(c), cancel with the explicit charge vertex renormalizations, shown in Figs. 3.5(h)–(j), as a consequence of the Ward identity associated with the minimalistic gauge transformation  $\psi \mapsto e^{ie\lambda(\tau)}\psi$ ,  $a \mapsto a - \partial_\tau \lambda(\tau)$  [282, 301]. The parameter  $c$  is an irrelevant parameter that is kept in order to regularize purely bosonic loops in the nematic sector, shown in Figs. 3.5(g,s). Note that such a regularization parameter is not required in the AIAO sector [301], since a cubic bosonic self-interaction is forbidden by the Ising symmetry of the AIAO order-parameter field  $\phi$ . In the above, we have expanded the flow equations assuming small  $c \ll 1$ . As will be shown in Sec. 3.5.3, this assumption is consistent with  $c$  being small at all fixed points of interest, namely, the LAB fixed point, the nematic fixed point, and the AIAO fixed point.

Following Ref. [282], we neglect the anomalous contribution  $\propto \sum_{i,j,k=1}^d \partial_i T_{ij} \partial_k T_{kj}$  in the nematic sector, allowed whenever the spatial dimension  $d$  agrees with  $\dim T$ , the dimension of the nematic tensor field  $T_{ij} = \sum_{a=1}^{(d-1)(d+2)/2} \varphi_a \Lambda_{a,ij}$ .

For simplicity, the flow equations for the nematic Yukawa coupling  $g_2^2$ , the self-interaction coupling  $\lambda$ , the regularization parameter  $c$ , and the nematic tuning parameter  $r_2$  have further been expanded to lowest order in the anisotropy parameter  $\delta$  around the isotropic case  $\delta = 0$ . This is sufficient to describe the nematic sector, see Sec. 3.5.3. For consistency, *before* any rescalings of coupling constants, also the involved anomalous dimensions  $\eta_\psi$  and  $\eta_\varphi$  and the dynamical exponent  $z$  have to be expanded to their leading orders in  $\delta$  in these cases,  $\eta_\psi^{(0)} = \eta_\psi|_{\delta=0}$ ,  $\eta_\varphi^{(0)} = \eta_\varphi|_{\delta=0}$ , and  $z^{(0)} = z|_{\delta=0}$ . Note, however, that anisotropy-dependent coupling reparametrizations, such as the above rescaling  $(e^2, g_1^2) \mapsto (1 - \delta^2)(e^2, g_1^2)$ , can still lead to crucial anisotropy dependencies. On the level of the  $4 - \epsilon$  expansion, this happens explicitly through two vertex corrections in the flow equation of  $g_2^2$  now involving a factor of  $1 - \delta^2$ , see Eq. (3.22), and implicitly in the flow equations of  $g_2^2$ ,  $\lambda$ , and  $c$  through the reparametrized anomalous dimension  $\eta_\psi^{(0)}$  involving a factor of  $1 - \delta^2$ , see Eq. (3.27). In general, the theory space is spanned by 13 different parameters, namely the anisotropy parameter  $\delta$ , the effective charge  $e$ , the AIAO Yukawa coupling  $g_1$ , the AIAO tuning parameter  $r_1$ , two nematic Yukawa couplings  $g_{2,t}$  and  $g_{2,e}$  (associated with the  $T_{2g}$  and  $E_g$  components of the nematic tensor order parameter), three cubic nematic self-interactions  $\lambda_{ttt}$ ,  $\lambda_{tte}$ , and  $\lambda_{eee}$ , two regularization parameters  $c_t$  and  $c_e$ , and two nematic tuning parameters  $r_{2,t}$  and  $r_{2,e}$ . Note that  $T_{2g} \otimes E_g \otimes E_g = 2T_{1g} \oplus 2T_{2g}$  [382], which forbids terms like  $\varphi_1 \varphi_4 \varphi_5$  in the action. This implies the absence of a fourth cubic

self-interaction governed by a coupling  $\lambda_{\text{tee}}$ . Expanding the flow in the nematic sector around the isotropic limit allows us to restrict the parameter space to an eight-dimensional subspace spanned by  $\delta$ ,  $e$ ,  $g_1$ ,  $r_1$ ,  $g_2$ ,  $\lambda$ ,  $c$ , and  $r_2$ . The cost of this simplification is the introduction of an additional free parameter  $c_1$  in the flow of the nematic Yukawa coupling  $g_2$ . However, the contribution involving  $c_1$  is merely a biquadratic term,  $\propto g_1^2 g_2^2$ . As such, we can already anticipate that its impact on the flow will likely be quantitative rather than qualitative. Indeed, as will be shown in Sec. 3.5.3, both the location as well as the scaling exponents of the LAB fixed point, the nematic fixed point, and the AIAO fixed point remain entirely unaffected by the value of  $c_1$ . We therefore expect that disregarding the distinction between the  $T_{2g}$  and  $E_g$  components of the nematic tensor order parameter will not affect the universal physics. For the numerical integration of the flow equations presented in Sec. 3.5.4, we choose  $c_1 \equiv 1/5$ , which represents the weighted average of the  $T_{2g}$  and  $E_g$  contributions  $c_{1t} = +1$  and  $c_{1e} = -1$ .

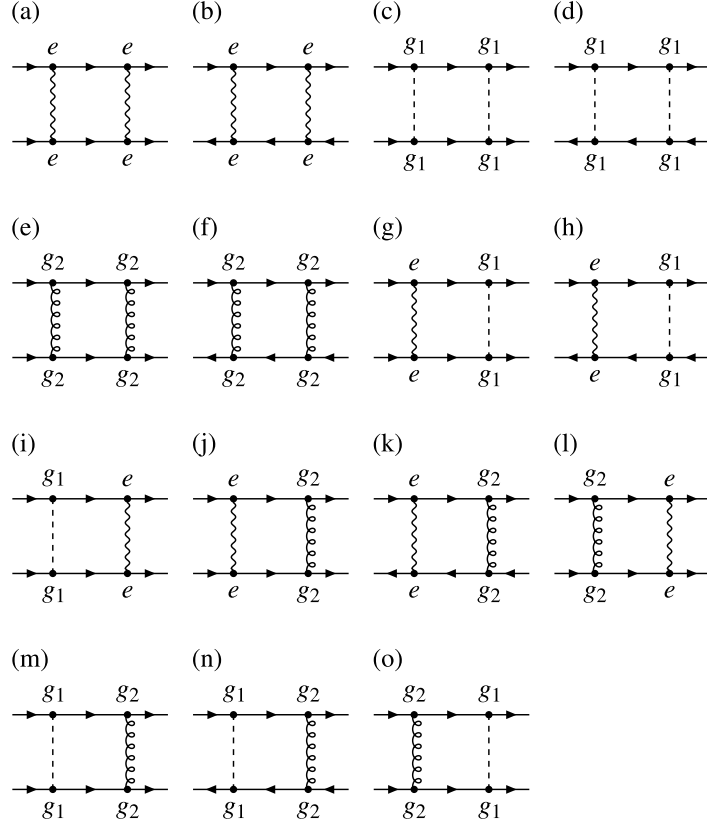
The flow equations presented here generalize those previously derived for the individual AIAO [301] and nematic [282] sectors, reducing to these in the respective limits.

### Dynamical bosonization scheme

In three spatial dimensions, it is no longer sufficient to solely work with the flow equations arising from the  $4 - \epsilon$  expansion. Additionally, we need to take into account four-fermion box diagrams, which generate further four-fermion interactions, not present at the microscopic scale. These are shown at the one-loop order in Fig. 3.6. Although these contributions can be safely neglected for small  $\epsilon = 4 - d \ll 1$ , four-fermion contributions to the nematic channel turn out to be crucial in  $2 < d < 4$  [282]. Neglecting these contributions would mask the fixed-point collision and annihilation of the LAB fixed point with the nematic fixed point when lowering the number of quadratic band touching points  $N$  below a critical flavor number  $N_c$ . Using dynamical bosonization in the nematic sector, renormalization-group-generated four-fermion interactions can be directly incorporated into the flow of the nematic Yukawa coupling  $g_2^2$  [282, 374–376, 383]. To illustrate this, consider the renormalized action

$$\begin{aligned}
S_{\text{FB}}^< = & \int_{\omega, \vec{q}} \frac{1}{2} \varphi_a (r_2 + \Delta r_2) \varphi_a + \int_{\omega_1, \vec{q}_1} \int_{\omega_2, \vec{q}_2} \frac{g_2 + \Delta g_2}{\sqrt{N}} \varphi_a \psi_i^\dagger \gamma_a \psi_i \\
& + \int_{\omega_1, \vec{q}_1} \int_{\omega_2, \vec{q}_2} \int_{\omega_3, \vec{q}_3} \frac{\Delta G_2}{N} (\psi_i^\dagger \gamma_a \psi_i)^2 + \dots,
\end{aligned} \tag{3.33}$$

where  $\int_{\omega, \vec{q}} := \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d}$  and the ellipsis denotes all other, tree level or renormalization-group-generated, terms. For notational simplicity, we assume summation convention over repeated indices here. The renormalizations  $\Delta r_2, \Delta g_2, \Delta G_2 = \mathcal{O}(\ln b)$  contribute to the nematic order-parameter mass, the nematic Yukawa vertex, and the four-fermion coupling in the nematic channel, respectively. Note that although the four-fermion interaction is absent in the ultraviolet, it gets generated under a renormalization group



**Figure 3.6.:** Complete set of four-fermion box diagrams at the one-loop order. In the dynamical bosonization scheme, diagrams (a)–(f) and (m)–(o) contribute to the flow of the nematic Yukawa coupling  $g_2$ . In contrast, diagrams (g)–(i) and (j)–(l) do not contribute to the nematic channel when adding the individual contributions. Figure reprinted from Ref. [283].

transformation. Shifting the nematic fields as  $\varphi_a \mapsto \varphi_a + \frac{\Delta w}{\sqrt{N}} \psi_i^\dagger \gamma_a \psi_i$  results in the shifted renormalized action

$$\begin{aligned}
 S_{\text{FB}}^{\leq} \mapsto & \int_{\omega, \vec{q}} \frac{1}{2} \varphi_a (r_2 + \Delta r_2) \varphi_a + \int_{\omega_1, \vec{q}_1} \int_{\omega_2, \vec{q}_2} \frac{g_2 + \Delta g_2 + r_2 \Delta w}{\sqrt{N}} \varphi_a \psi_i^\dagger \gamma_a \psi_i \\
 & + \int_{\omega_1, \vec{q}_1} \int_{\omega_2, \vec{q}_2} \int_{\omega_3, \vec{q}_3} \frac{\Delta G_2 + g_2 \Delta w}{N} (\psi_i^\dagger \gamma_a \psi_i)^2 + \dots,
 \end{aligned} \tag{3.34}$$

assuming  $\Delta w = \mathcal{O}(\ln b)$ . Choosing  $\Delta w = -\Delta G_2/g_2$  eliminates the four-fermion interaction at the cost of the additional Yukawa vertex renormalization  $\propto -r_2 \Delta G_2/g_2$ . In other words, the renormalization-group-generated four-fermion interaction can be incorporated through a Hubbard-Stratonovich transformation after each renormalization group step, resulting in a modified flow equation for the Yukawa coupling. Within the dynamical bosonization scheme, the flow equation of the nematic Yukawa coupling  $g_2^2$  takes the form

$$\begin{aligned}
 \frac{dg_2^2}{d \ln b} = & (6 - d - z^{(0)} - \eta_\varphi^{(0)} - 2\eta_\psi^{(0)}) g_2^2 + \frac{12}{5N} \frac{g_2^4}{1 + r_2} \\
 & + \frac{4}{5N} (1 - \delta^2) g_2^2 e^2 - \frac{4c_1}{5N} (1 - \delta^2) \frac{g_1^2 g_2^2}{1 + r_1} \\
 & + \frac{1}{10N} (1 - \delta^2)^2 r_2 e^4 + \frac{1}{10N} (1 - \delta^2)^2 \frac{r_2 g_1^4}{(1 + r_1)^2}
 \end{aligned}$$

$$+ \frac{13}{10N} \frac{r_2 g_2^4}{(1+r_2)^2} - \frac{c_2}{N} (1-\delta^2) \frac{r_2 g_1^2 g_2^2}{(1+r_1)(1+r_2)}, \quad (3.35)$$

where the first two lines stem from the  $4 - \epsilon$  expansion above and the last two lines are the contributions arising from the dynamical bosonization scheme. Restricting to the eight-dimensional parameter space required us to introduce another free parameter  $c_2$ . As with the previously encountered parameter  $c_1$ , the contribution corresponding to  $c_2$  is also simply biquadratic,  $\propto g_1^2 g_2^2$ . We show in Sec. 3.5.3 that the location and the scaling exponents of the LAB fixed point, the nematic fixed point, and the AIAO fixed point are independent of both  $c_1$  and  $c_2$ . For the numerical integration of the flow equations in Sec. 3.5.4, we choose, in analogy to the procedure for  $c_1$ , the weighted average  $c_2 \equiv 2/5$  of the  $T_{2g}$  and  $E_g$  contributions  $c_{2t} = 0$  and  $c_{2e} = 1$ . The flow equations in the dynamical bosonization scheme reduce to the corresponding equations from Ref. [282] in the nematic sector when setting  $g_1 = \delta = 0$ .

### 3.5.3. Fixed-point structure

In this subsection, we provide details of the different renormalization group fixed points. In particular, we show that the LAB fixed point collides (annihilates) with the nematic fixed point at (below) a critical number of quadratic band touching points  $N_c > 1$ . Moreover, we demonstrate that the AIAO fixed point does not take part in the collision and annihilation process, but remains in the physical space of real couplings for any number of quadratic band touching points  $N \in \mathbb{N}$ .

To observe the fixed-point annihilation at finite coupling strengths, it is convenient to reparametrize the AIAO Yukawa coupling  $g_1^2 \mapsto g_1^2/(-\delta)$  and the corresponding AIAO tuning parameter  $r_1 \mapsto r_1/(-\delta)$ . Additionally, we introduce the new tuning variables

$$\alpha_1 := \frac{r_1}{1+r_1}, \quad \alpha_2 := \frac{r_2}{1+r_2}, \quad (3.36)$$

which map the AIAO and nematic tuning parameters  $r_1$  and  $r_2$  for  $r_1, r_2 \in [0, \infty)$  onto the unit interval,  $\alpha_1, \alpha_2 \in [0, 1)$ . The fixed-point equations for the eight parameters  $\delta, e^2, g_1^2, \alpha_1, g_2^2, \lambda, c, \alpha_2$  can be solved analytically in the limit  $N \rightarrow \infty$ . Starting from the large- $N$  solutions, we use a Newton-Raphson method in order to determine the fixed-point values and the corresponding scaling or critical exponents numerically at finite  $N$ . For sufficiently large  $N$ , we identify a unique interacting infrared stable fixed point, which we associate with the disordered LAB phase [328–330]. The corresponding fixed-point values for different values of  $N$ , together with the associated scaling exponents, are given in Table 3.1. In addition, as expected from the mean-field analysis, cf. Sec. 3.4, we further find two quantum critical fixed points associated with nematic [282, 331] and AIAO [301, 326, 327] instabilities. Their corresponding fixed-point values and critical exponents are given in Tables 3.2 and 3.3, respectively.

**Table 3.1.:** LAB fixed point: Fixed-point values and corresponding scaling exponents in  $d = 3$  spatial dimensions from dynamical bosonization for different numbers of quadratic band touching points  $N \geq N_c = 1.856$ . Below the critical number  $N_c$ , the LAB fixed point has annihilated with the nematic fixed point and is no longer present in the physical space of real couplings.

| $N$      | $\delta_\star$ | $e_\star^2$ | $g_{1\star}^2$  | $\alpha_{1\star}$ | $g_{2\star}^2$  | $-\lambda_\star$           | $c_\star$      | $\alpha_{2\star}$ | $2-z$ | $\eta_a$ | $\eta_\phi$ | $\eta_\varphi$ | $\omega_1$ | $\omega_2$ |
|----------|----------------|-------------|-----------------|-------------------|-----------------|----------------------------|----------------|-------------------|-------|----------|-------------|----------------|------------|------------|
| $N_c$    | 0.000          | 0.874       | 0.732           | 1.000             | 1.446           | 0.040                      | 0.162          | 0.927             | 0.126 | 0.874    | 1.004       | 1.818          | 0.036      | 0.000      |
| 2        | 0.000          | 0.882       | 0.737           | 1.000             | 1.494           | 0.040                      | 0.164          | 0.952             | 0.118 | 0.882    | 1.011       | 1.878          | 0.034      | 0.269      |
| 3        | 0.000          | 0.918       | 0.738           | 1.000             | 1.542           | 0.040                      | 0.163          | 0.976             | 0.082 | 0.918    | 1.012       | 1.938          | 0.023      | 0.624      |
| 4        | 0.000          | 0.937       | 0.737           | 1.000             | 1.556           | 0.040                      | 0.162          | 0.983             | 0.063 | 0.937    | 1.011       | 1.956          | 0.018      | 0.740      |
| 5        | 0.000          | 0.949       | 0.736           | 1.000             | 1.564           | 0.040                      | 0.162          | 0.987             | 0.051 | 0.949    | 1.009       | 1.966          | 0.014      | 0.800      |
| 10       | 0.000          | 0.974       | 0.733           | 1.000             | 1.578           | 0.040                      | 0.161          | 0.994             | 0.026 | 0.974    | 1.005       | 1.984          | 0.007      | 0.908      |
| 25       | 0.000          | 0.989       | 0.731           | 1.000             | 1.586           | 0.040                      | 0.160          | 0.997             | 0.011 | 0.989    | 1.002       | 1.994          | 0.003      | 0.965      |
| 100      | 0.000          | 0.997       | 0.730           | 1.000             | 1.590           | 0.040                      | 0.159          | 0.999             | 0.003 | 0.997    | 1.001       | 1.998          | 0.001      | 0.991      |
| $\infty$ | 0              | 1           | $\frac{35}{48}$ | 1                 | $\frac{35}{22}$ | $\sqrt{\frac{105}{13310}}$ | $\frac{7}{44}$ | 1                 | 0     | 1        | 1           | 2              | 0          | 1          |

As visible from Tables 3.1 and 3.2, the nematic fixed point and the LAB fixed point are located in the physical space of real couplings for  $N \geq N_c = 1.8555070\dots$  (rounded to three decimal places for better readability in what follows). At  $N = N_c$ , the LAB fixed point collides with the nematic fixed point and for even lower numbers of quadratic band touching points  $N < N_c$ , the two fixed points take complex values with finite imaginary parts. This result is consistent with the earlier analyses [281, 282, 334]. For the LAB fixed point, we show the dynamical exponent  $z$ , which is related to the fermion anomalous dimension as  $\eta_\psi = 2 - z$ , the photon anomalous dimension  $\eta_a$ , the anomalous dimension  $\eta_\phi$  of the AIAO order-parameter field, the anomalous dimension  $\eta_\varphi$  of the nematic order parameter, as well as the two least negative eigenvalues of the stability matrix at the LAB fixed point,  $\omega_1$  and  $\omega_2$ , corresponding to leading and next-to-leading corrections to scaling. We have labelled  $\omega_1$  and  $\omega_2$  such that  $\omega_1 < \omega_2$  at large  $N$ , and smoothly connected to this limit for finite  $N$ . The scaling exponent  $\omega_1$  reflects the fact that the anisotropy  $\delta$  becomes an exactly marginal operator in the limit of infinitely many quadratic band touching points  $N \rightarrow \infty$ . The scaling exponent  $\omega_2 \rightarrow 0^+$  for  $N \rightarrow N_c^+$  signals the collision of the LAB fixed point with the nematic fixed point. For the nematic fixed point, we analogously show  $z$ ,  $\eta_a$ ,  $\eta_\phi$ , and  $\eta_\varphi$ , as well as the inverse correlation-length exponent  $1/\nu$ , corresponding to the unique positive eigenvalue of the stability matrix. We note that the AIAO order parameter is non-critical at the nematic quantum critical point, and the corresponding anomalous dimension  $\eta_\phi$  at the nematic fixed point determines the scaling of the AIAO correlation function only after subtracting the momentum-independent contribution. The numerical values of  $z$ ,  $\eta_a$ ,  $\eta_\varphi$ ,  $\omega_2$ , and  $1/\nu$  in Tables 3.1 and 3.2 precisely agree with the corresponding values given in Ref. [282]. This is a consequence of the fact that AIAO fluctuations are suppressed by the infinite mass of the AIAO order-parameter field  $\phi$  at the LAB and nematic fixed points, as indicated by the corresponding  $\alpha_{1\star} = 1$ .

As shown in Table 3.3, the AIAO fixed point exists for any number of quadratic band touching points  $N \in \mathbb{N}$  and, therefore, is not involved in any fixed-point annihilations.

**Table 3.2.:** Nematic fixed point: Fixed-point values and corresponding critical exponents in  $d = 3$  spatial dimensions from dynamical bosonization for different numbers of quadratic band touching points  $N \geq N_c = 1.856$ . Below the critical number  $N_c$ , the nematic fixed point has annihilated with the LAB fixed point and is no longer present in the physical space of real couplings.

| $N$      | $\delta_\star$ | $e_\star^2$ | $g_{1\star}^2$  | $\alpha_{1\star}$ | $g_{2\star}^2$  | $-\lambda_\star$           | $c_\star$      | $\alpha_{2\star}$ | $2-z$ | $\eta_a$ | $\eta_\phi$ | $\eta_\varphi$ | $1/\nu$ |
|----------|----------------|-------------|-----------------|-------------------|-----------------|----------------------------|----------------|-------------------|-------|----------|-------------|----------------|---------|
| $N_c$    | 0.000          | 0.874       | 0.732           | 1.000             | 1.446           | 0.040                      | 0.162          | 0.927             | 0.126 | 0.874    | 1.004       | 1.818          | 0.000   |
| 2        | 0.000          | 0.882       | 0.725           | 1.000             | 1.378           | 0.039                      | 0.158          | 0.892             | 0.118 | 0.882    | 0.994       | 1.732          | 0.260   |
| 3        | 0.000          | 0.916       | 0.717           | 1.000             | 1.215           | 0.038                      | 0.145          | 0.805             | 0.084 | 0.916    | 0.983       | 1.528          | 0.577   |
| 4        | 0.000          | 0.934       | 0.716           | 1.000             | 1.125           | 0.037                      | 0.137          | 0.755             | 0.066 | 0.934    | 0.981       | 1.414          | 0.681   |
| 5        | 0.000          | 0.946       | 0.716           | 1.000             | 1.065           | 0.037                      | 0.132          | 0.721             | 0.054 | 0.946    | 0.982       | 1.339          | 0.740   |
| 10       | 0.000          | 0.971       | 0.720           | 1.000             | 0.934           | 0.036                      | 0.120          | 0.644             | 0.029 | 0.971    | 0.988       | 1.174          | 0.862   |
| 25       | 0.000          | 0.988       | 0.725           | 1.000             | 0.851           | 0.035                      | 0.112          | 0.594             | 0.012 | 0.988    | 0.994       | 1.070          | 0.943   |
| 100      | 0.000          | 0.997       | 0.728           | 1.000             | 0.809           | 0.035                      | 0.108          | 0.569             | 0.003 | 0.997    | 0.999       | 1.017          | 0.985   |
| $\infty$ | 0              | 1           | $\frac{35}{48}$ | 1                 | $\frac{35}{44}$ | $\sqrt{\frac{105}{85184}}$ | $\frac{7}{66}$ | $\frac{14}{25}$   | 0     | 1        | 1           | 1              | 1       |

**Table 3.3.:** AIAO fixed point: Fixed-point value and exact critical exponents in  $d = 3$  spatial dimensions. The AIAO fixed point is present in the physical space of real couplings for any number of quadratic band touching points  $N \in \mathbb{N}$ .

| $N$ | $\delta_\star$ | $e_\star^2$    | $g_{1\star}^2$ | $\alpha_{1\star}$ | $g_{2\star}^2$  | $-\lambda_\star$           | $c_\star$      | $\alpha_{2\star}$ | $2-z$ | $\eta_a$ | $\eta_\phi$ | $\eta_\varphi$ | $1/\nu$ |
|-----|----------------|----------------|----------------|-------------------|-----------------|----------------------------|----------------|-------------------|-------|----------|-------------|----------------|---------|
| all | -1             | $\frac{9}{16}$ | $\frac{9}{16}$ | 0                 | $\frac{35}{44}$ | $\sqrt{\frac{105}{85184}}$ | $\frac{7}{66}$ | 1                 | 0     | 1        | 1           | 1              | 1       |

The fixed point is characterized by maximal anisotropy  $\delta_\star = -1$ , which has the consequence that the location of the fixed point and the corresponding critical exponents can be computed analytically. As argued in Sec. 3.5.5, the AIAO critical exponents are expected to be exact. Our results for  $z$ ,  $\eta_a$ ,  $\eta_\phi$ , and  $1/\nu$  also precisely agree with the earlier calculations [301, 326].

We have explicitly verified that locations and exponents of all three fixed points are independent from the parameters  $c_1$  and  $c_2$  introduced in Sec. 3.5.2. This implies that also our result for the critical number  $N_c$  at which the fixed-point collision takes place does depend on neither  $c_1$  nor  $c_2$ . This justifies a posteriori the earlier introduced restriction to an eight-dimensional theory space.

### 3.5.4. Integration of flow equations

In this subsection, we give details on the procedure for obtaining Figs. 3.3(b)–(d) and 3.4 from numerically integrating out the flow equations, and provide further comments on the universality class associated with the continuous order-to-order transition.

#### Zero-temperature phase diagram

To begin with, we eliminate the Yukawa couplings  $g_1^2$  and  $g_2^2$  by introducing effective couplings  $G_1$  and  $G_2$ , corresponding to the microscopic couplings in the original four-

fermion action, Eq. (3.2), as

$$G_1 = \frac{g_1^2}{r_1} \quad \text{and} \quad G_2 = \frac{g_2^2}{r_2}, \quad (3.37)$$

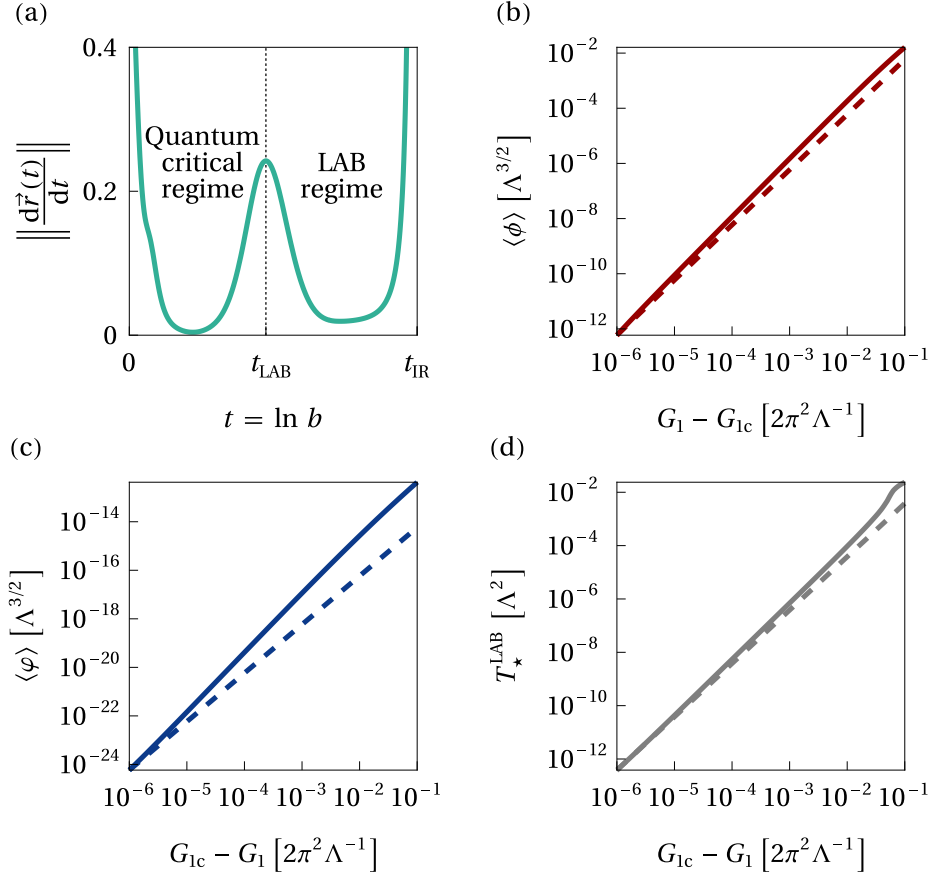
cf. Eq. (3.7). To obtain the zero-temperature phase diagram, we numerically integrate out the flow equations for the eight parameters  $\delta, e^2, G_1, \alpha_1, G_2, \lambda, c, \alpha_2$  in  $d = 3$  spatial dimensions, using varying initial effective interaction strengths in the AIAO (nematic) channel  $G_1$  ( $G_2$ ), and fixed  $(\delta, 2e^2/(3\pi^2\Lambda), \alpha_1, \lambda/\sqrt{2\pi^2\Lambda}, c\Lambda^2, \alpha_2)_{\text{UV}} = (-\frac{1}{2}, 1, \frac{1}{10}, -\sqrt{\frac{105}{85184}}, \frac{7}{66}, \frac{14}{25})$  at the ultraviolet scale. In order to allow for an easier comparison with the mean-field results and other studies, we have chosen to work with dimensionful quantities  $e^2, G_1, G_2, \lambda$ , and  $c$  in this subsection. The initial values of the effective charge  $e^2$ , the cubic nematic self-interaction  $\lambda$ , the regularization parameter  $c$ , and the nematic mass parameter  $\alpha_2$  have been chosen according to the location of the nematic fixed point in the limit of infinitely many quadratic band touching points  $N \rightarrow \infty$ . The factors of  $2/(3\pi^2)$  and  $1/\sqrt{2\pi^2}$  in the definition of the starting values for  $e^2$  and  $\lambda$  arise from the coupling rescaling for  $\delta = -1/2$ , as given after Eq. (3.32). While the location of the phase boundaries depends on the choice of the initial values, our universal results, such as the occurrence of a continuous order-to-order transition is independent of this choice. Depending on the chosen initial conditions, for  $N > N_c$ , we find a flow towards either of three different regimes: For small  $G_1$  and  $G_2$ , the parameters flow towards the LAB fixed-point values. In particular, this implies that  $\alpha_1$  and  $\alpha_2$  remain positive at all renormalization group scales. This regime is associated with the LAB phase. For positive  $G_1$  beyond a certain finite threshold, we find that  $\alpha_1$  becomes negative at some renormalization group scale. Negative  $\alpha_1$  corresponds to a negative curvature of the effective potential in the AIAO sector. Consequently, we associate the flow of  $\alpha_1$  toward negative values with the onset of AIAO order. For positive  $G_2$  beyond a certain finite threshold, on the other hand, we find that  $\alpha_2$  becomes negative at some renormalization group scale, which we associate with the onset of nematic order. In order to estimate the values of the order parameters in the two different ordered phases, we assume that the flow *away* from the quantum critical fixed point is dominated by simple dimensional scaling. This assumption only modifies nonuniversal properties of the observables; in particular, it does neither change the order of the transition nor the values of the corresponding critical exponents [94, 383, 384]. Under this assumption, the values of the order parameters can be computed from the renormalization group time  $t_{\text{IR}} = \ln b_{\text{IR}}$ , at which the associated parameters  $\alpha_1$  and  $\alpha_2$ , respectively, turn negative. In particular, in the AIAO phase, the order parameter  $\langle\phi\rangle$  is related to  $t_{\text{IR}}$  via  $\langle\phi\rangle \propto \exp\left(-\frac{d+z-2+\eta_\phi}{2}t_{\text{IR}}\right) = \exp(-2t_{\text{IR}})$ , since  $z = 2$  and  $\eta_\phi = 1$  at the AIAO quantum critical fixed point for all numbers of quadratic band touching points  $N \in \mathbb{N}$  and in  $d = 3$  spatial dimensions. In the nematic phase, the corresponding order parameter  $\langle\varphi\rangle$  can be computed analogously; however, one has to be careful which dynamical exponent and anomalous dimension to incorporate: For  $N = 10$  quadratic band touching points, we identify  $\langle\varphi\rangle \propto \exp\left(-\frac{d+z-2+\eta_\varphi}{2}t_{\text{IR}}\right) = \exp(-2.072t_{\text{IR}})$ , with  $z = 1.971$  and  $\eta_\varphi = 1.174$  at the nematic fixed point, see Table 3.2. Likewise, for  $N = 2$  quadratic band touching

points, we employ again  $\langle\varphi\rangle \propto \exp\left(-\frac{d+z-2+\eta_\varphi}{2}t_{\text{IR}}\right) = \exp(-2.307t_{\text{IR}})$ , since  $z = 1.882$  and  $\eta_\varphi = 1.732$  at the nematic fixed point, see Table 3.2. However, for the case  $N = 1$ , and therewith after the fixed-point annihilation, we identify  $\langle\varphi\rangle \propto \exp\left(-\frac{d+z-2+\eta_\phi}{2}t_{\text{IR}}\right) = \exp(-2t_{\text{IR}})$ , with  $z = 2$  and  $\eta_\phi = 1$  from the annihilation-surviving AIAO fixed point, see Table 3.3, which governs the transition into the nematic phase. Using these prescriptions, we arrive at Figs. 3.3(b)–(d) and 3.4(a) from numerical integration of the flow equations.

### Finite-temperature phase diagram

In order to obtain the finite-temperature phase diagram for  $N = 1$ , shown in Fig. 3.4(b), we again assume that the flow away from the quantum critical fixed point is dominated by simple dimensional scaling. Up to prefactors of order one, we can then estimate the critical temperature on the AIAO side of the transition as  $T_c^{\text{AIAO}} \propto \exp(-zt_{\text{IR}}) = \exp(-2t_{\text{IR}})$ , where  $t_{\text{IR}} = \ln b_{\text{IR}}$  denotes the renormalization group time at which the parameter  $\alpha_1$  turns negative. Similarly, on the nematic side of the transition, we can estimate the nematic critical temperature as  $T_c^{\text{nem}} \propto \exp(-2t_{\text{IR}})$ , with  $t_{\text{IR}}$  denoting the renormalization group time at which the parameter  $\alpha_2$  turns negative. For  $N = 1$ , the complex LAB fixed point is still reasonably close to the physical space of real couplings. This results in a regime with logarithmically slow flow and LAB-like behavior in a sizable finite-temperature regime [281, 282]. The corresponding crossover temperature  $T_\star^{\text{LAB}}$  between the nonuniversal high-temperature regime and the LAB-like regime at intermediate temperatures can be estimated via  $T_\star^{\text{LAB}} \propto \exp(-2t_{\text{LAB}})$ , where  $t_{\text{LAB}}$  corresponds to the renormalization group time at which the flow enters the slow-flow regime associated with the complex LAB fixed point. A natural way to estimate  $t_{\text{LAB}}$  is to analyze the dimensionless vector of flow parameters  $\vec{r}(t) := (\delta, e^2/[2\pi^2(1-\delta^2)\Lambda], G_1\Lambda/[2\pi^2(1-\delta^2)], \alpha_1, G_2\Lambda/(2\pi^2), \lambda/\sqrt{2\pi^2\Lambda}, c\Lambda^2, \alpha_2)(t)$  as a function of renormalization group time  $t$ . Figure 3.7(a) shows the corresponding flow speed  $\left\|\frac{d\vec{r}(t)}{dt}\right\|$  as a function of  $t \in [0, t_{\text{IR}}]$ . We observe two regimes in which the flow becomes slow. Comparing the flow parameters  $\vec{r}(t)$  with the values of the different fixed points, we identify the first (second) regime of slow flow to arise from the vicinity to the AIAO (complex LAB) fixed point. The first minimum in  $\left\|\frac{d\vec{r}(t)}{dt}\right\|$  thus corresponds to the quantum critical regime, while the second minimum corresponds to the LAB regime. The local maximum in  $\left\|\frac{d\vec{r}(t)}{dt}\right\|$  in between the two minima at intermediate time  $t_{\text{LAB}}$  thus corresponds to the crossover between the two regimes.

In sum, on the nematic side of the transition, we find a sizable intermediate-temperature regime between the critical temperature  $T_c^{\text{nem}}$  and the crossover temperature  $T_\star^{\text{LAB}}$ . On the AIAO side, there is only a single minimum in the flow speed  $\left\|\frac{d\vec{r}(t)}{dt}\right\|$ , associated with the quantum critical regime, indicating the absence of an additional intermediate-temperature regime, in agreement with Fig. 3.4(b).



**Figure 3.7.:** (a) Flow speed  $\left\| \frac{d\vec{r}(t)}{dt} \right\|$  as a function of renormalization group time  $t \in [0, t_{\text{IR}}]$  for  $N = 1$  and a set of initial values of the flow parameters on the nematic side of the transition, namely  $\vec{r}(t = 0) = \left( -\frac{1}{2}, 1, \frac{G_{1c}}{3\pi^2/(2\Lambda)} - \frac{4}{3} \cdot 10^{-5}, \frac{1}{10}, 0, -\sqrt{\frac{105}{85184}}, \frac{7}{66}, \frac{14}{25} \right)$ , with  $G_{1c} = 0.752 \dots \cdot \frac{3\pi^2}{2\Lambda}$ . The speed displays two regimes of slow flow, indicated by the two minima of  $\left\| \frac{d\vec{r}(t)}{dt} \right\|$ . The first (second) regime arises from the vicinity to the AIAO (complex LAB) fixed point. The local maximum at intermediate  $t_{\text{LAB}}$  corresponds to the crossover between the two regimes. (b) AIAO order parameter  $\langle \phi \rangle$  as a function of  $G_1 - G_{1c}$  in double logarithmic plot on the AIAO side of the transition for  $N = 1$ . All other initial flow parameters have been chosen as in (a). Dashed line indicates the theoretically expected slope  $\nu z = 2$ . (c) Same as (b), but for the nematic order parameter  $\langle \varphi \rangle$  as a function of  $G_{1c} - G_1$  on the nematic side of the transition. The exceptionally small value of the nematic order parameter is a result of the complex LAB fixed point still being reasonable close to the physical space of real couplings at  $N \lesssim N_c = 1.856$ , resulting in a slow flow and a sizable intermediate-temperature LAB-like regime. (d) Same as (c), but for the crossover temperature  $T_{\star}^{\text{LAB}}$  between the nonuniversal high-temperature regime and the intermediate-temperature LAB-like regime on the nematic side of the transition. Figure reprinted from Ref. [283].

### Universality class

As a cross-check of our numerical analysis, we demonstrate that the observables computed by integrating out the flow equations lead to the same critical exponents as those obtained from our analytical solutions of the set of fixed-point equations and the eigenvalue problem of the corresponding stability matrix, cf. Sec. 3.5.3. Since the continuous order-to-order transition is governed by the AIAO fixed point, we expect the correlation-length exponent  $\nu = 1$  and the anomalous dimensions of the AIAO and nematic order-parameter fields  $\eta_\phi = \eta_\varphi = 1$ . The hyperscaling assumption implies for the order-parameter exponents

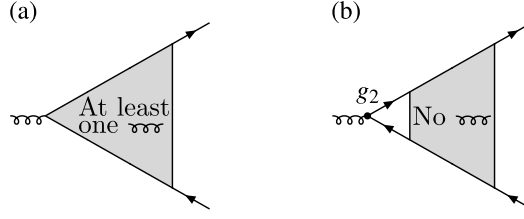
$\beta_\phi = \beta_\varphi = 2$  on the two sides of the transition. Further, we expect the dynamical critical exponent  $z = 2$ . Figure 3.7(b) displays the AIAO order parameter as a function of the tuning parameter  $G_1 - G_{1c}$  on the AIAO side of the transition. Near the critical point, the values obtained from numerical integration (solid line) approach the expected behavior with slope  $\beta_\phi = 2$  (dashed line), as expected from the fixed-point analysis. Similarly, Fig. 3.7(c) displays the nematic order parameter as a function of  $G_{1c} - G_1$  on the nematic side of the transition. While corrections to scaling appear to be more significant in this case, the leading behavior is still consistent with the expected slope  $\beta_\phi = 2$ . We attribute the sizable scaling corrections to the slow flow arising from the fixed-point annihilation and the marginal anisotropy parameter  $\delta$ , which leads to quasiuniversal behavior over several length scales [301]. Lastly, Fig. 3.7(d) depicts the crossover temperature  $T_\star^{\text{LAB}}$  as a function of  $G_{1c} - G_1$  on the nematic side of the transition. Again, the numerical result (solid line) in the vicinity of the critical point is consistent with the expectation from the fixed-point analysis (dashed line), namely  $T_\star^{\text{LAB}} \propto (G_{1c} - G_1)^{\nu z}$  with  $\nu z = 2$ .

Let us end this subsection with two general comments on the universality class of an order-to-order quantum phase transition caused by the proposed fixed-point-annihilation mechanism. First, it is of course no coincidence that the two correlation-length exponents  $\nu_\phi$  and  $\nu_\varphi$  associated with the AIAO and nematic correlation functions align in our theory,  $\nu = \nu_\phi = \nu_\varphi$ . This has to hold for any order-to-order quantum phase transition arising from fixed-point annihilation as a consequence of the presence of a single divergent length scale at the annihilation-surviving quantum critical fixed point. Second, all other pairs of critical exponents associated with the two different order parameters may in general take on different values. The fact that this does not happen in the present case can be understood as a consequence of the maximal anisotropy  $\delta_\star = -1$  at the AIAO fixed point, leading to  $\eta_\phi = \eta_\varphi = 1$ , which, together with the hyperscaling assumption, implies that all pairs of exponents agree.

### 3.5.5. Higher-loop corrections

In this subsection, we demonstrate that the critical exponents at the AIAO fixed point obtained at the one-loop order do not receive higher-loop corrections, and are thus expected to be exact. This is ultimately another consequence of the maximal anisotropy  $\delta_\star = -1$  at the AIAO fixed point. Our argument is independent of the number of quadratic band touching points  $N$  and therefore holds for all  $N \in \mathbb{N}$ .

As argued in Ref. [301] for  $G_2 = 0$ , higher-loop corrections in the AIAO and charge sector are fully suppressed at the AIAO fixed point. This can be understood as a consequence of the fact that the AIAO fixed point is located at maximal anisotropy  $\delta_\star = -1$ . Here, it can be seen from the fact that the fixed point values for  $e^2$  and  $g_1^2$  are finite *after* performing the anisotropy-dependent reparametrization  $(e^2, g_1^2) \mapsto (1 - \delta^2)(e^2, g_1^2)$ , cf. Sec. 3.5.2. After this rescaling, every higher-loop diagram now comes with a strictly positive power of  $(1 - \delta^2)$ . As a result, all higher-loop corrections are strongly suppressed in the vicinity of the AIAO



**Figure 3.8.:** Every loop contribution to the nematic Yukawa vertex falls into one of two distinct classes, (a) the class of diagrams which involve at least one internal nematic boson and (b) the class of diagrams which do not involve any internal nematic boson. While class (a) does not contribute at the AIAO fixed point due to infinite nematic boson mass  $r_{2*} = \infty$ , class (b) vanishes at the AIAO fixed point due to maximal spatial anisotropy  $\delta_* = -1$ . Figure reprinted from Ref. [283].

fixed point at anisotropy  $\delta_* = -1$ , rendering the one-loop flow equations asymptotically exact, see Ref. [301] for details and Refs. [341, 385] for a different model with analogous behavior.

In our extended model for finite  $G_1$  and  $G_2$ , the AIAO fixed point remains to be located at maximal anisotropy  $\delta_* = -1$  and additionally lies now at infinite nematic boson mass  $r_{2*} \rightarrow \infty$ , cf. Table 3.3. These two fixed-point values suppress all higher-loop diagrams contributing to the flow equations relevant for the critical exponents. For instance, all higher-loop corrections to the flow of the anisotropy  $\delta$ , to the effective charge  $e^2$ , and to the AIAO Yukawa coupling  $g_1^2$  are either strongly suppressed by strictly positive powers of  $(1 - \delta^2)$  (these are the already known contributions arising in the theory for  $G_1 = 0$ ) and/or by strictly positive powers of  $1/(1 + r_2)$  (these are new contributions with at least one internal nematic boson) in the vicinity of the AIAO fixed point. Consequently, the anomalous dimension of the fermion  $\eta_\psi = 0$ , the dynamical exponent  $z = 2 - \eta_\psi = 2$ , the anomalous dimension of the photon  $\eta_a = 4 - d$ , and the anomalous dimension the AIAO boson  $\eta_\phi = 4 - d$  are exact critical exponents of the AIAO fixed point. A similar argument can be made for the new flow equation of the nematic Yukawa coupling  $g_2^2$  here, since all loop-correction fall again into two distinct classes, as depicted in Fig. 3.8, which both vanish at the AIAO fixed point. The first class of diagrams consists of all loop-corrections involving at least one internal nematic boson. Since internal nematic bosons give rise to the appearance of strictly positive powers of  $1/(1 + r_2)$ , the whole first class of diagrams is suppressed in the vicinity of the AIAO fixed point located at infinite nematic mass  $r_{2*} = \infty$ . Any higher-loop order correction that is not of the above form involves only internal fermion, photon, and/or AIAO boson propagators and is, following the argument presented in Ref. [301], expected to be suppressed by a strictly positive power of  $(1 - \delta^2)$ . Therefore, also the anomalous dimension of the nematic order-parameter field  $\eta_\varphi = 4 - d$  is expected to be exact at the AIAO fixed point. The only exponent left to think about is the correlation-length exponent  $\nu$ . Note that the AIAO and nematic propagator display the same exponent  $\nu = \nu_\phi = \nu_\varphi$ , since there is only one divergent length scale present at the AIAO fixed point, as demonstrated in the previous subsection. Analogously to the flow equation of the anisotropy  $\delta$ , the effective charge  $e^2$ , and the AIAO Yukawa coupling  $g_1^2$ , higher-loop corrections to the flow of the AIAO mass  $r_1$  are either strongly suppressed by strictly positive powers of  $(1 - \delta^2)$  and/or by strictly positive powers of  $1/(1 + r_2)$  in

the vicinity of the AIAO fixed point. As a consequence, the stability matrix is of block triangular form with  $1/\nu = 2 - \eta_\phi = d - 2$  as the only (strictly) positive eigenvalue.

### 3.5.6. Fermionic renormalization group flow

In this last subsection, we show that the dynamically bosonized flow equations of our model using Hubbard-Stratonovich fields  $\phi$  and  $\varphi$ , as obtained in Sec. 3.5.2, reduce, in the limit of infinite order-parameter masses (point-like limit),  $\alpha_1, \alpha_2 \rightarrow 1$ , to those of a purely fermionic model, obtained without introducing order-parameter fields explicitly. The latter is defined by the action

$$S_F = \int d\tau d^3x \left\{ \psi^\dagger \left[ \partial_\tau + \sum_{a=1}^5 (1 + s_a \delta) d_a (-i\nabla) \gamma_a + \frac{ie}{\sqrt{N}} a \right] \psi + \frac{1}{2} (\nabla a)^2 - \frac{G_1}{2N} (\psi^\dagger \gamma_{45} \psi)^2 - \frac{G_2}{2N} \sum_{a=1}^5 (\psi^\dagger \gamma_a \psi)^2 \right\}, \quad (3.38)$$

and has been studied previously in Refs. [281, 282, 327]. The goal of the comparison is twofold. First, we want to demonstrate that our calculations are consistent with the fermionic ones presented in Refs. [281, 282, 327]. Second, we want build some intuition on fermionic flows with respect to the proposed fixed-point-annihilation mechanism, which prepares us for the examples discussed below. We note, however, that the limit of infinite order-parameter masses is in principle insufficient to compute order parameters in symmetry-broken phases, as required for explicitly demonstrating the proposed order-to-order transition. This is because fermionic renormalization group flows diverge in the case of spontaneous symmetry breaking.

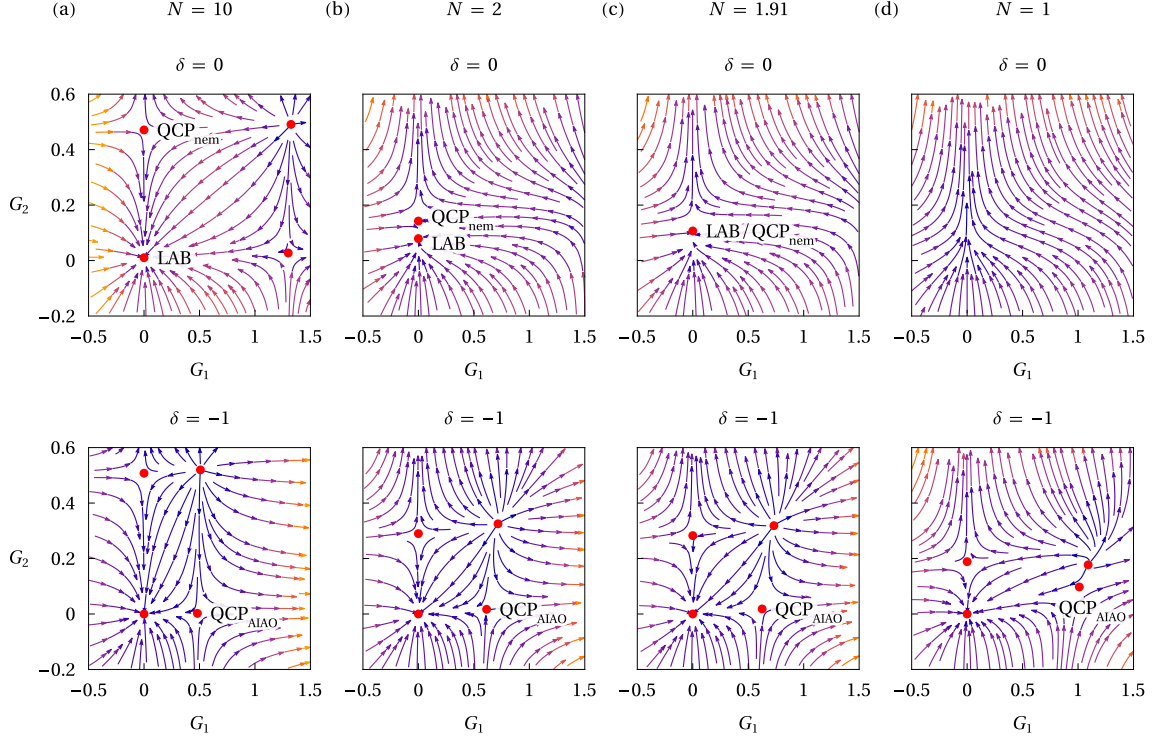
We start from the flow equations in the dynamical bosonization scheme in terms of the couplings  $\delta$ ,  $e^2$ ,  $G_1$ ,  $\alpha_1$ ,  $G_2$ ,  $\lambda$ ,  $c$ , and  $\alpha_2$ . In the limit of infinite order-parameter masses,  $\alpha_1, \alpha_2 \rightarrow 1$ , the flow equations for  $\delta$ ,  $e^2$ ,  $G_1$ , and  $G_2$  become independent of  $\lambda$  and  $c$ . In this limit, we find

$$\frac{d\delta}{d \ln b} = -\eta_\psi \delta + \frac{2}{15N} (1 - \delta^2) [(1 + \delta) f_{1t} - (1 - \delta) f_{1e}] e^2, \quad (3.39)$$

$$\frac{de^2}{d \ln b} = \left( z + 2 - d - \eta_a + \frac{2\delta}{1 - \delta^2} \frac{d\delta}{d \ln b} \right) e^2, \quad (3.40)$$

$$\begin{aligned} \frac{dG_1}{d \ln b} &= (z - d) G_1 + \frac{2}{5N} (1 - \delta) (1 - \delta^2) f_{2e} e^2 G_1 \\ &\quad - \frac{2(1 - 2N)}{5N} (1 - \delta) f_{2e} G_1^2 - \frac{2}{5N} (1 - \delta) f_{2e} G_1 G_2, \end{aligned} \quad (3.41)$$

$$\begin{aligned} \frac{dG_2}{d \ln b} &= (z^{(0)} - d) G_2 + \frac{37 + 16N}{10N} G_2^2 + \frac{4}{5N} (1 - \delta^2) e^2 G_2 \\ &\quad + \frac{1}{10N} (1 - \delta^2)^2 e^4 - \frac{4c_1 + 5c_2}{5N} G_1 G_2 + \frac{1}{10N} G_1^2, \end{aligned} \quad (3.42)$$



**Figure 3.9.:** (a) Renormalization group flow diagram in  $d = 3$  spatial dimensions and for  $N = 10$  in the point-like limit  $\alpha_1, \alpha_2 \rightarrow 1$  (Eqs. (3.39)–(3.44)) in the plane spanned by  $G_1$  and  $G_2$ . We fix  $(\delta, e^2) = (0, 15N/(4 + 15N))$  (top panel) and  $(\delta, e^2) = (-1, 9/16)$  (bottom panel), respectively. Arrows indicate flow towards infrared. Besides the infrared stable fixed point associated with the disordered LAB phase (LAB), there are two quantum critical points associated with instabilities towards nematic ( $\text{QCP}_{\text{nem}}$ ) and AIAO ( $\text{QCP}_{\text{AIAO}}$ ) orders, respectively. (b) Same as (a), but for  $N = 2$ , illustrating that the LAB and  $\text{QCP}_{\text{nem}}$  fixed point approach each other for decreasing  $N$ . (c) Same as (a), but for  $N = 1.91$ , illustrating the collision of the LAB and  $\text{QCP}_{\text{nem}}$  fixed points at a critical  $N > 1$ . (d) Same as (a), but for  $N = 1$ , illustrating the runaway flow in the isotropic plane  $\delta = 0$  (top panel) and the persistence of  $\text{QCP}_{\text{AIAO}}$  in the plane  $\delta = -1$  (bottom panel). Figure reprinted from Ref. [283].

with the anomalous dimensions

$$\eta_\psi = \frac{2}{15N}(1 - \delta^2)[(1 + \delta)f_{1t} + (1 - \delta)f_{1e}]e^2, \quad (3.43)$$

$$\eta_a = e^2 f_{e^2}, \quad (3.44)$$

the full anisotropy-dependent dynamical exponent  $z = 2 - \eta_\psi$ , and the corresponding form  $z^{(0)} = 2 - \frac{4}{15N}(1 - \delta^2)e^2$  in the isotropic limit  $\delta \rightarrow 0$ , relevant for the flow in the nematic channel  $G_2$ . To arrive at the above flow equations, we have rescaled the AIAO order-parameter mass in the dynamically bosonized flow equations as  $r_1 \mapsto (1 - \delta^2)r_1$ , such that  $G_1 = g_1^2/r_1$  remains finite at the AIAO quantum critical point. The dimensionless couplings occurring in Eqs. (3.39)–(3.44) are then related to the dimensionful couplings of the microscopic action, Eq. (3.38), via the rescaling  $e^2 \mapsto 2\pi^2\Lambda^{4-d}(1 - \delta^2)e^2$ ,  $G_1 \mapsto 2\pi^2\Lambda^{2-d}G_1$ , and  $G_2 \mapsto 2\pi^2\Lambda^{2-d}G_2$ . For  $N = 1$ , the flow equations for the anisotropy  $\delta$ , the charge  $e^2$ , and the anomalous dimensions  $\eta_\psi$  and  $\eta_a$  agree with the previous calculation [327]. For all  $N \in \mathbb{N}$ , but  $\delta = 0$ , the flow equations for the charge  $e^2$  and the anomalous dimensions  $\eta_\psi$  and  $\eta_a$  agree with those of Refs. [281, 282]. The flow of  $G_2$  also agrees, with the ex-

ception of slight differences in the prefactors of the terms  $\propto G_1 G_2$  and  $\propto G_1^2$ . These can be attributed to the fact that we have performed the dynamical bosonization scheme only in the nematic channel, leading to a Fierz-incomplete calculation. This also explains the (minor) differences in the flow of  $G_1$  in comparison with the corresponding calculation in Refs. [281, 282].

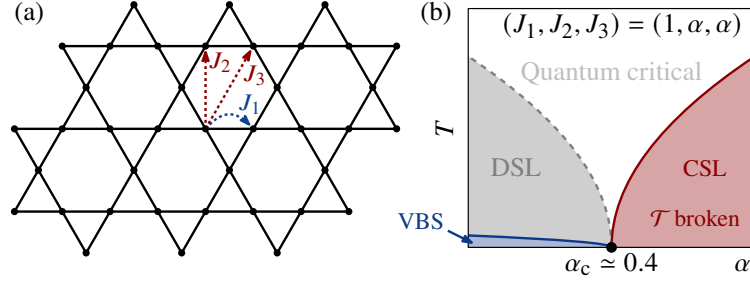
For  $N > 1.91$ , the fermionic flow equations feature a fully infrared stable fixed point, associated with the LAB phase, as well as two quantum critical fixed points, associated with the instabilities towards nematic and AIAO orders, respectively. We note that the AIAO fixed point in the point-like limit, in contrast to our results in the dynamical bosonization scheme (Sec. 3.5.3), exhibits an additional marginally relevant direction, associated with the flow of the anisotropy parameter  $\delta$ . This is a known artifact of the fermionic renormalization group flow that can be attributed to the negligence of AIAO order-parameter fluctuations in the point-like limit [301]. For  $N = 1.91$ , the infrared stable LAB fixed point and the nematic quantum critical point collide and subsequently disappear into the complex plane for  $N < 1.91$ , in qualitative agreement with the fixed-point structure obtained from the dynamically bosonized renormalization group flow, discussed in Sec. 3.5.3, as well as the earlier works [281, 282, 327]. In agreement with the dynamically bosonized renormalization group flow, the fixed-point collision takes place in the isotropic  $G_1$ - $G_2$  plane defined by  $\delta = 0$  and finite effective charge  $e^2 = \frac{15N}{4+15N}$ . The flow in this plane is depicted for different values of  $N$  in Fig. 3.9 (top row), illustrating the fixed-point collision. Although all fixed points in this subsection are analytically accessible, we refrain from stating their expressions here, as they are rather lengthy.

Importantly, the AIAO fixed point does not take part in the collision and annihilation of the LAB fixed point with the nematic fixed point, but remains in the physical space of real couplings for all  $N \in \mathbb{N}$ , in agreement with our results in the dynamical bosonization scheme, cf. Table 3.3. The AIAO fixed point is located in the plane defined by anisotropy  $\delta = -1$  and effective charge  $e^2 = \frac{9}{16}$ . The flow in this plane is depicted for different values of  $N$  in Fig. 3.9 (bottom row), illustrating the fact that the AIAO fixed point persists for all  $N \in \mathbb{N}$ . We conclude, that, at least in the case of Luttinger semimetals, the presented fixed-point-annihilation mechanism is also available in the purely fermionic formulation.

## 3.6. Kagome quantum magnets

### 3.6.1. Model

Although the proposed mechanism for a continuous order-to-order transition does not depend on the formation of fractionalized quasiparticles, it is possible to emerge also in conjunction with fractionalization phenomena. To illustrate this point, consider the



**Figure 3.10.:** (a) Kagome lattice, indicating first ( $J_1$ ), second ( $J_2$ ), and third ( $J_3$ ) neighbors. (b) Schematic finite-temperature phase diagram of kagome Heisenberg model as a function of  $\alpha = J_2/J_1 = J_3/J_1$ , from renormalization group analysis of effective QED<sub>3</sub>-Gross-Neveu model. The gapped chiral spin liquid (CSL) breaks time reversal. Solid (dashed) lines denote phase transitions (crossovers). The disordered gapless Dirac spin liquid (DSL) is unstable at low temperatures towards a valence bond solid (VBS), leaving behind a continuous zero-temperature transition between distinct orders. Figure reprinted from Ref. [283].

extended Heisenberg model on the kagome lattice, defined by the Hamiltonian

$$\mathcal{H} = J_1 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle\langle ij \rangle\rangle} \vec{S}_i \cdot \vec{S}_j + J_3 \sum_{\langle\langle\langle ij \rangle\rangle\rangle} \vec{S}_i \cdot \vec{S}_j. \quad (3.45)$$

Here,  $\vec{S}_i$  represents a spin-1/2 at site  $i$ , while  $\langle ij \rangle$ ,  $\langle\langle ij \rangle\rangle$ , and  $\langle\langle\langle ij \rangle\rangle\rangle$  correspond to first-, second-, and third-neighbor bonds, respectively, with  $J_1$ ,  $J_2$ , and  $J_3$  denoting the associated exchange couplings, see Fig. 3.10(a). The model describes magnetic Mott insulators crystallizing in a layered kagome structure, such as herbertsmithite  $\text{ZnCu}_3(\text{OH})_6\text{Cl}_2$  [386], kapellasite  $\text{Cu}_3\text{Zn}(\text{OH})_6\text{Cl}_2$  [387], or  $\text{YCu}_3(\text{OH})_{6.5}\text{Br}_{2.5}$  [388, 389]. For dominant antiferromagnetic  $J_1 > 0$ , numerical simulations at finite temperature or on finite lattices observe no evidence for any long-range-ordered state, suggesting the formation of a quantum spin liquid [390–398]. The nature of this state has been a matter of significant debate, with recent simulations indicating a gapless U(1) Dirac spin liquid [394–396]. For increasing second- and third-neighbor interactions  $J_2$  and  $J_3$ , a transition to a gapped chiral spin liquid, characterized by broken time reversal symmetry with finite scalar spin chirality  $\langle \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k) \rangle$ , has been found [399–403]. For instance, fixing the second- and third-nearest-neighbor couplings as  $J_2/J_1 = J_3/J_1 = \alpha$  with antiferromagnetic next-neighbor coupling  $J_1 > 0$ , the transition occurs at  $\alpha_c \simeq 0.4$  [399]. The transition has been argued to be continuous [399] and described by the QED<sub>3</sub>-Gross-Neveu model with  $N = 2$  four-component Dirac fields [243–245, 251, 394, 404, 405]. The QED<sub>3</sub>-Gross-Neveu model describes fermionic excitations  $\psi$  and  $\psi^\dagger$  arising from fractionalization of the spin degrees of freedom, interacting via an emergent U(1) gauge field  $a_\mu$  and additional local four-fermion interactions,

$$S_{\text{QED}_3\text{-GN}} = \int d^3x \left[ \bar{\psi}_i \gamma_\mu \left( \partial_\mu - \frac{ie}{\sqrt{N}} a_\mu \right) \psi_i + \frac{1}{4} f_{\mu\nu}^2 - \frac{G_1}{2N} (\bar{\psi}_i \gamma_{45} \psi_i)^2 - \frac{G_2}{2N} (\bar{\psi}_i \gamma_\mu \psi_i)^2 \right]. \quad (3.46)$$

Here,  $\psi_i$  and  $\bar{\psi}_i := \psi_i^\dagger \gamma_3$  denote  $N = 2$  flavors of four-component Dirac spinors in  $(2 + 1)$ -dimensional Euclidean space-time,  $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$  is the  $U(1)$  field-strength tensor associated with the  $U(1)$  gauge field  $a_\mu$ , the  $4 \times 4$  matrices  $\gamma_\mu$  satisfy the Clifford algebra, and  $\gamma_{45} := i\gamma_4\gamma_5$  is the Hermitian product of the two chiral gamma matrices in the reducible four-component spinor representation. For notational simplicity, we have assumed the summation convention over repeated flavor indices  $i = 1, \dots, N$  and space-time indices  $\mu = 1, 2, 3$ . Among others, the model features  $\mathbb{Z}_2$  time reversal and  $SU(2N)$  flavor symmetry [243].

### 3.6.2. Renormalization group analysis

In  $2+1$  dimensions, the gauge coupling  $e^2$  has positive mass dimension and flows to a finite value in the infrared. The four-fermion couplings  $G_1$  and  $G_2$  are power-counting irrelevant, but can be generated by the finite gauge coupling. On the level of mean-field theory, the coupling  $G_1$ , if tuned beyond a certain finite threshold, induces a transition between the Dirac spin liquid for  $G_1 < G_{1c}$  and the chiral spin liquid for  $G_1 > G_{1c}$ . The latter is characterized by a finite vacuum expectation value of the fermion bilinear  $\langle \bar{\psi}_i \gamma_{45} \psi_i \rangle \neq 0$ , which breaks time reversal symmetry [406]. The four-fermion coupling  $G_1$  can thus be understood to parameterize the second- and third-nearest-neighbor interactions  $J_2 = J_3$  in the microscopic model. The coupling  $G_2$  is generated by the renormalization group flow and is therefore included from the outset. Indeed, treating  $G_1$  and  $G_2$  on equal footing is essential for identifying the continuous order-to-order transition.

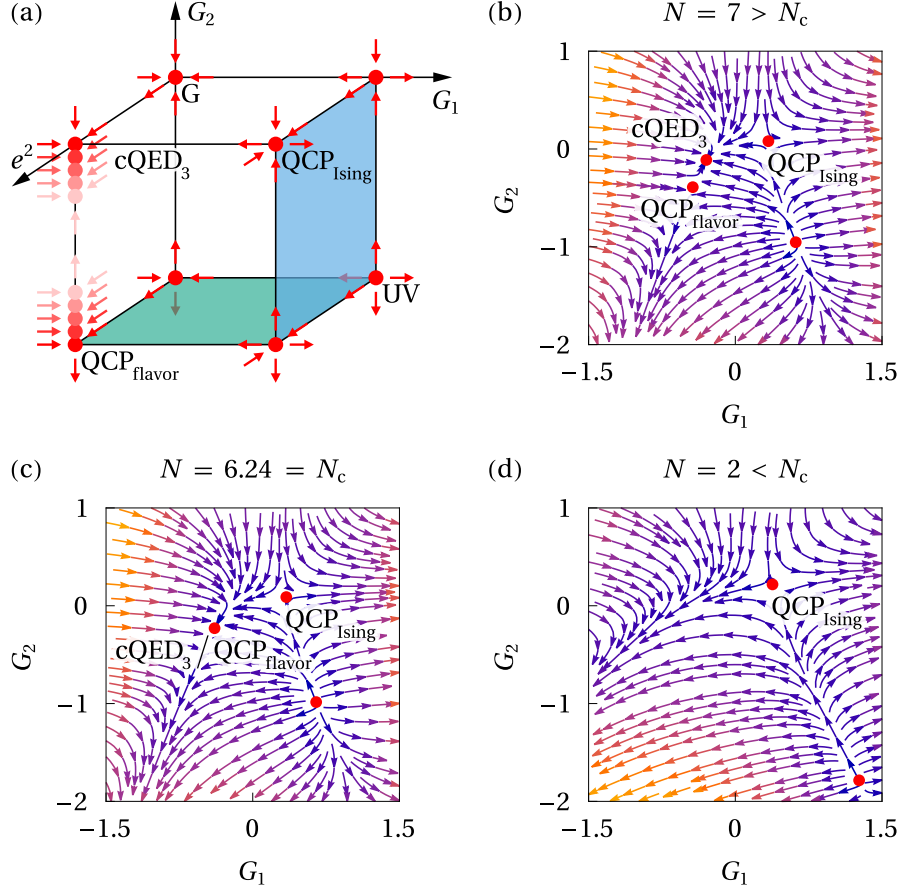
At one-loop order, the flow equations for the QED<sub>3</sub>-Gross-Neveu model read [243, 285]

$$\frac{de^2}{d \ln b} = (1 - \eta_a) e^2, \quad (3.47)$$

$$\frac{dG_1}{d \ln b} = -G_1 + \frac{16}{3N} e^2 G_1 + \frac{8}{N} e^2 G_2 - \frac{4}{N} e^4 + \frac{2(2N-1)}{N} G_1^2 - \frac{4}{N} G_2^2 - \frac{6}{N} G_1 G_2, \quad (3.48)$$

$$\frac{dG_2}{d \ln b} = -G_2 + \frac{8}{3N} e^2 G_1 - \frac{2(2N+1)}{3N} G_2^2 - \frac{2}{N} G_1 G_2, \quad (3.49)$$

with gauge field anomalous dimension  $\eta_a = \frac{4}{3}e^2$ . Above, we employed the rescaling  $e^2 \mapsto 2\pi^2 \Lambda e^2$ ,  $G_1 \mapsto 2\pi^2 \Lambda^{-1} G_1$ , and  $G_2 \mapsto 2\pi^2 \Lambda^{-1} G_2$ , where  $\Lambda$  denotes the ultraviolet cutoff. Figure 3.11(a) illustrates the fixed-point structure of the model. For sufficiently large  $N$ , the infrared attractive plane  $e^2 = e_\star^2 = 3/4$  hosts a fully infrared stable fixed point (cQED<sub>3</sub>), associated with the conformal phase of three-dimensional quantum electrodynamics [284–286, 407]. In the context of the extended Heisenberg spin-1/2 model on the kagome lattice, the latter can be understood as a  $U(1)$  Dirac spin liquid [394–396]. In addition, the plane features two quantum critical fixed points, associated with instabilities towards Ising  $\mathbb{Z}_2$  time-reversal-symmetry-breaking (QCP<sub>Ising</sub>) and  $SU(2N)$  flavor-symmetry-breaking (QCP<sub>flavor</sub>) ground states, respectively [284, 286]. On the kagome lattice, the former can be understood as chiral spin liquid [394, 404], while the latter is expected to realize valence-bond-solid order [408].



**Figure 3.11.:** (a) Schematic fixed-point structure of the QED<sub>3</sub>-Gross-Neveu model in the space spanned by the couplings  $e^2$ ,  $G_1$ , and  $G_2$ . Arrows indicate flow towards infrared. For sufficiently large  $N$ , the infrared attractive plane  $e^2 = e_\star^2$  hosts a fully infrared stable fixed point (cQED<sub>3</sub>), associated with the conformal phase of three-dimensional quantum electrodynamics, representing the U(1) Dirac spin liquid state in the kagome Heisenberg spin-1/2 model. In addition, the plane features two quantum critical fixed points, associated with instabilities towards  $\mathbb{Z}_2$  time-reversal-symmetry-breaking (QCP<sub>Ising</sub>) and SU(2N) flavor-symmetry-breaking (QCP<sub>flavor</sub>) ground states, respectively. On the kagome lattice, the former can be understood as chiral spin liquid, while the latter is expected to realize valence-bond-solid order. The critical manifold with respect to time reversal symmetry breaking (flavor symmetry breaking) is indicated in blue (green). For decreasing  $N$ , cQED<sub>3</sub> and QCP<sub>flavor</sub> approach each other and eventually collide at a critical flavor number  $N_c$ . For orientation, we have labelled the Gaussian fixed point (G) and the ultraviolet stable fixed point (UV). (b) Renormalization group flow diagram for  $N = 7$  in the plane  $e^2 = e_\star^2$  spanned by the four-fermion couplings  $G_1$  and  $G_2$ , indicating the locations of the infrared stable cQED<sub>3</sub> fixed point and the quantum critical QCP<sub>Ising</sub> and QCP<sub>flavor</sub> fixed points. (c) Same as (b), but for the critical flavor number  $N = N_c = 6.24$ , indicating the fixed-point collision. (d) Same as (b), but for  $N = 2$ , relevant for the extended Heisenberg spin-1/2 model on the kagome lattice. The absence of the cQED<sub>3</sub> fixed point implies that the U(1) Dirac spin liquid is unstable and the QCP<sub>Ising</sub> fixed point governs a direct continuous transition between valence-bond-solid order and a time-reversal-symmetry-breaking chiral spin liquid. Figure reprinted from Ref. [283].

For decreasing  $N$ , the infrared stable cQED<sub>3</sub> fixed point and the quantum critical QCP<sub>flavor</sub> fixed point approach each other and eventually collide at a critical flavor number  $N_c$ . This is illustrated in Figs. 3.11(b)–(d), which shows the renormalization group flow in the  $G_1$ - $G_2$  plane for fixed  $e^2 = e_\star^2$  and different values of  $N$  above, at, and below the critical flavor number  $N_c$ . For  $N < N_c$ , the cQED<sub>3</sub> and QCP<sub>flavor</sub> fixed points are located in the complex plane, leaving behind a runaway flow in the space of real couplings. Importantly,

the fixed point  $\text{QCP}_{\text{Ising}}$ , associated with the time-reversal-symmetry-breaking instability, remains unaffected by the collision and annihilation process. For  $N < N_c$ , it distinguishes between two different regimes characterized by runaway flows: Consider the line of starting values given by  $e^2 = e_\star^2$  and  $G_2 = 0$  at the ultraviolet scale. For  $G_1 > G_{1\star}$ , the flow diverges towards  $(G_1, G_2) \rightarrow (\infty, 0)$  at a finite renormalization group scale, indicating spontaneous time reversal symmetry breaking. For  $G_1 < G_{1\star}$ , on the other hand, the flow diverges towards  $(G_1, G_2) \rightarrow (-\infty, -\infty)$ , which has previously been interpreted as an instability towards flavor symmetry breaking [243, 286, 406]. This suggests that  $\text{QCP}_{\text{Ising}}$  governs for  $N < N_c$  a continuous quantum phase transition between two long-range-ordered states that break different symmetries. At the one-loop order, we find the critical flavor number as  $N_c = 6.24$ . Higher-loop contributions may reduce the number, but most analytical estimates [284, 286, 407, 409, 410] appear to suggest that the fixed-point collision and annihilation occurs *above* the physical number  $N = 2$  relevant for the kagome quantum magnets. We emphasize, however, that we are not aware of clear numerical evidence for the scenario in an unbiased simulation of the  $\text{QED}_3$ -Gross-Neveu model, such as Monte Carlo [411, 412]. If indeed  $N_c > 2$  in the  $\text{QED}_3$ -Gross-Neveu model, it implies for the extended Heisenberg spin-1/2 model on the kagome lattice that (1) the U(1) Dirac spin liquid becomes unstable at the lowest temperatures towards a valence-bond-solid-ordered ground state and (2) the quantum phase transition between the valence bond solid, stabilized for  $\alpha < 0.4$ , and the chiral spin liquid, stabilized for  $\alpha > 0.4$ , is continuous and governed by the  $\text{QCP}_{\text{Ising}}$  fixed point. The resulting schematic phase diagram is depicted in Fig. 3.10(b). An explicit computation of the order parameters across the order-to-order transition would require introducing the corresponding Hubbard-Stratonovich fields within the dynamical bosonization framework of Sec. 3.5, which we leave for future work.

### 3.7. Spin-boson models

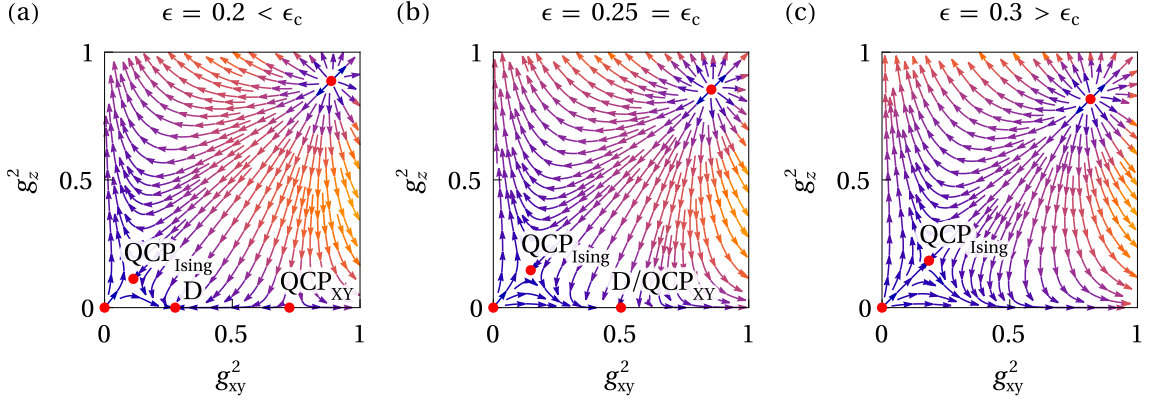
As an example for a model featuring a continuous order-to-order transition from fixed-point annihilation that is directly amenable to large-scale numerical simulations, consider the anisotropic spin-boson model, which describes a single spin  $\vec{S}$  in a fluctuating magnetic field  $\vec{h}$  with an XXZ anisotropy, defined by the Hamiltonian [413]

$$\mathcal{H}_{\text{spin-boson}} = g_{xy}(h^x S^x + h^y S^y) + g_z h^z S^z + \mathcal{H}_{\text{bulk}}(\vec{h}). \quad (3.50)$$

Here,  $g_{xy}$  and  $g_z$  correspond to in-plane and out-of-plane components of the  $g$  tensor, respectively, and the bulk Hamiltonian  $\mathcal{H}_{\text{bulk}}(\vec{h})$  is chosen such that the fluctuations of the magnetic field are Gaussian, with power-law correlation

$$\langle \mathcal{T}_\tau h^a(\tau) h^b(0) \rangle \propto \frac{\delta^{ab}}{|\tau|^{2-\epsilon}}, \quad (3.51)$$

where  $\mathcal{T}_\tau$  corresponds to the time-ordering operator in imaginary time  $\tau$ . The bath exponent  $\epsilon$  represents an external tuning parameter, and we consider the case of small positive



**Figure 3.12.:** (a) Renormalization group flow diagram of the anisotropic spin-boson model for  $\epsilon = 0.2$  in the plane spanned by  $g_{xy}^2$  and  $g_z^2$ . Arrows indicate flow towards infrared. For  $g_{xy}^2 < g_z^2$  at the ultraviolet scale, the parameters flow towards  $(g_{xy}^2, g_z^2) \rightarrow (0, +\infty)$ , associated with an Ising ordered phase. For  $g_{xy}^2$  above, but not too far from  $g_z^2$ , the flow is attracted by the fully infrared stable fixed point D, corresponding to an interacting disordered phase. For large  $g_{xy}^2$ , the parameters flow towards  $(g_{xy}^2, g_z^2) \rightarrow (+\infty, 0)$ , associated with an XY ordered phase. (b) Same as (a), but for  $\epsilon = \epsilon_c = 1/4$ , illustrating the collision of the quantum critical fixed point  $\text{QCP}_{XY}$  and the infrared stable fixed point D. (c) Same as (a), but for  $\epsilon = 0.3$ . For all values of  $g_{xy}^2 > g_z^2$ , the parameters flow towards  $(g_{xy}^2, g_z^2) \rightarrow (+\infty, 0)$ , associated with the  $xy$ -localized phase. The quantum critical fixed point  $\text{QCP}_{\text{Ising}}$  on the isotropic axis  $g_{xy}^2 = g_z^2$  governs a continuous transition between the two distinct orders. Figure reprinted from Ref. [283].

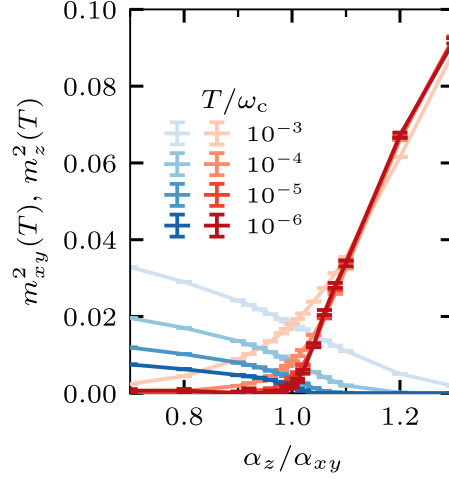
$\epsilon \in (0, 1/2)$ . The above spin-boson model may be realized as an effective description of a quantum impurity in a two-dimensional antiferromagnet in the vicinity of a quantum critical point [414–416].

The renormalization group flow equations for the two couplings  $g_{xy}$  and  $g_z$  up to two-loop order read [417, 418]

$$\frac{dg_{xy}^2}{d \ln b} = \epsilon g_{xy}^2 - g_{xy}^2(g_{xy}^2 + g_z^2) + g_{xy}^4(g_{xy}^2 + g_z^2), \quad (3.52)$$

$$\frac{dg_z^2}{d \ln b} = \epsilon g_z^2 - 2g_{xy}^2 g_z^2 + 2g_{xy}^2 g_z^4, \quad (3.53)$$

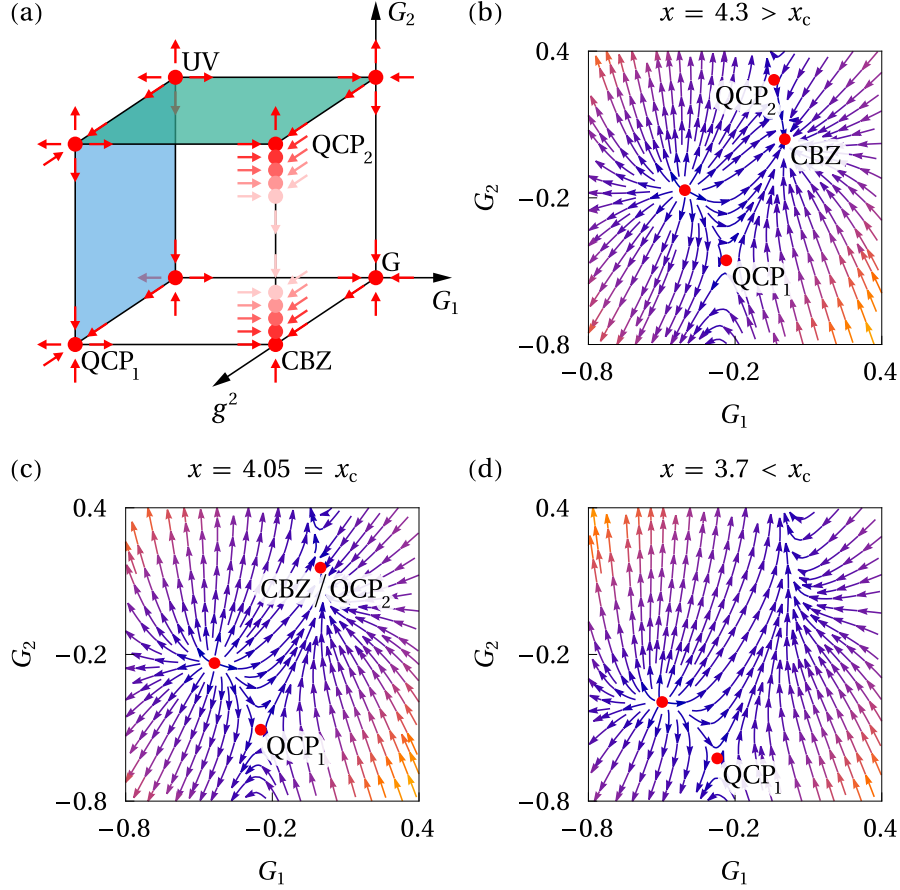
where we have rescaled  $g_{xy}^2 \mapsto \Lambda^\epsilon g_{xy}^2$  and  $g_z^2 \mapsto \Lambda^\epsilon g_z^2$ . Here,  $\Lambda$  denotes an arbitrary ultraviolet cutoff and  $\ln b$  is the renormalization group time, with  $\ln b = 0$  ( $\ln b \rightarrow \infty$ ) corresponding to the ultraviolet (infrared) limit. The flow is illustrated for different values of  $0 < \epsilon < 1/2$  in Fig. 3.12. For  $\epsilon > 0$ , both couplings  $g_{xy}^2$  and  $g_z^2$  are infrared relevant. For  $g_{xy}^2 < g_z^2$  at the ultraviolet scale, the parameters flow towards  $(g_{xy}^2, g_z^2) \rightarrow (0, +\infty)$ . This regime is associated with an Ising ordered phase, in which the spin forms local order with a finite moment along the  $z$  direction. The  $z$ -localized phase spontaneously breaks Ising  $\mathbb{Z}_2$  symmetry [290]. For  $g_{xy}^2 > g_z^2$  at the ultraviolet scale, the phase structure depends on the value of the external tuning parameter  $\epsilon$ . This is due to a fixed-point annihilation that takes place at a critical value of the tuning parameter  $\epsilon_c = 1/4$ . For  $\epsilon < \epsilon_c = 1/4$  and  $g_{xy}^2$  above, but not too far from  $g_z^2$ , the flow is attracted by a fully infrared stable fixed point, corresponding to an interacting disordered phase (D), see Fig. 3.12(a). For large  $g_{xy}^2$ , the parameters flow towards  $(g_{xy}^2, g_z^2) \rightarrow (+\infty, 0)$ . This regime is associated with an XY



**Figure 3.13.:** Local moment  $m_{xy}^2(T)$  ( $m_z^2(T)$ ) in the  $xy$ -plane ( $z$  direction) depicted in blue (red) as a function of the coupling  $\alpha_z$  for fixed  $\alpha_{xy} = 0.317$  and different values of the temperature  $T$  in units of the ultraviolet cutoff  $\omega_c$ , see Ref. [290] for details. The chosen bath exponent  $s = 0.7$  corresponds to  $\epsilon = 1 - s = 0.3$ , cf. Fig. 3.12(c), and the couplings  $\alpha_i$  can be convert to the couplings  $g_i$  used in this section via  $2\alpha_i = g_i^2$ . The displayed results from exact wormhole quantum Monte Carlo simulations confirm a continuous quantum phase transition between the  $z$ -localized and  $xy$ -localized phases. Although the fixed-point annihilation in the numerical calculations is reasonably far away ( $\epsilon_c = 0.2294(1)$ ), the local moment  $m_{xy}^2$  is still significantly smaller than  $m_z^2$  at low temperatures, as expected from the proposed fixed-point-annihilation mechanism. Note that also finite-temperature corrections are much stronger on the XY side of the transition. Figure adapted from Ref. [290].

ordered phase, in which the spin forms local order with a finite moment perpendicular to the  $z$  direction. For  $\epsilon = \epsilon_c$ , the fixed point D collides with the quantum critical fixed point associated with the onset of XY order, see Fig. 3.12(b). Now for  $\epsilon > \epsilon_c$ , the parameters flow towards  $(g_{xy}^2, g_z^2) \rightarrow (+\infty, 0)$  for all values of  $g_{xy}^2 > g_z^2$ . Most importantly, on the isotropic axis  $g_{xy}^2 = g_z^2$ , there is a quantum critical fixed point  $\text{QCP}_{\text{Ising}}$  that governs a continuous transition between Ising order and the disordered phase for  $0 < \epsilon < \epsilon_c$  and between the Ising order and XY order phase for  $\epsilon_c < \epsilon < 1/2$ . For  $\epsilon$  above, but not too far from  $\epsilon_c$ , the two-loop flow equations thus suggest a continuous order-to-order transition governed by the  $\text{QCP}_{\text{Ising}}$  fixed point.

The qualitative picture indicated by the perturbative calculation has recently been confirmed in large-scale numerical calculations [289, 290]; see also Refs. [287, 419] for recent analytical approaches based on large- $S$  expansions. The numerical calculations show that the true critical value of the control parameter  $\epsilon$ , at which the quantum critical fixed point  $\text{QCP}_{\text{XY}}$  and the infrared stable fixed point D collide, is  $\epsilon_c = 0.2294(1)$ , and thus close to the perturbative estimate. Interestingly, the numerical data for the transition between the Ising and XY ordered phases, reprinted in Fig. 3.13, confirm our theoretical predictions for a continuous order-to-order transition from fixed-point annihilation: First, on the XY side of the transition, the corresponding order parameter is significantly suppressed in comparison with its counterpart on the Ising side. Second, finite-temperature effects are significantly more enhanced on the XY side of the transition. Both properties can be understood as direct consequence of the annihilation between the  $\text{QCP}_{\text{XY}}$  and D fixed

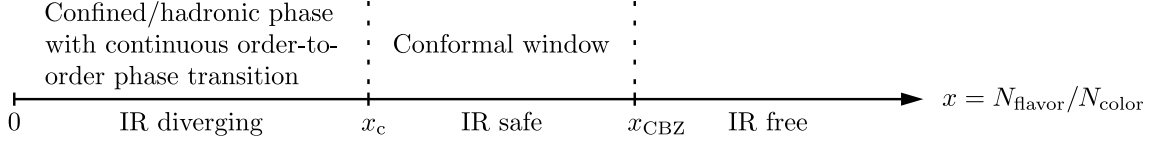


**Figure 3.14.:** (a) Schematic fixed-point structure of quantum chromodynamics in  $3 + 1$  space-time dimensions, with additional four-fermion interactions, in the Veneziano limit  $N_{\text{flavor}}, N_{\text{color}} \rightarrow \infty$ , with fixed ratio  $x = N_{\text{flavor}}/N_{\text{color}} < 11/2$ , in the space spanned by the couplings  $g^2$ ,  $G_1$ , and  $G_2$ . Arrows indicate flow towards infrared. For  $x$  below, but not too far from  $11/2$ , the infrared attractive plane  $g^2 = g_\star^2(G_1) = (11 - 2x - 3xG_1)/(13x - 34)$  hosts the fully infrared stable Caswell-Banks-Zaks fixed point (CBZ), as well as two quantum critical fixed points (QCP<sub>1</sub> and QCP<sub>2</sub>). The corresponding critical manifolds are indicated in blue and green, respectively. For decreasing  $x$ , CBZ and QCP<sub>2</sub> approach each other and eventually collide at a critical  $x_c$ . For orientation, we have labelled the Gaussian fixed point (G) and the ultraviolet stable fixed point (UV). (b) Renormalization group flow diagram for  $x = 4.3$  in the plane  $g^2 = g_\star^2$  spanned by the four-fermion couplings  $G_1$  and  $G_2$ , indicating the locations of the infrared stable CBZ fixed point and the quantum critical fixed points QCP<sub>1</sub> and QCP<sub>2</sub>. (c) Same as (b), but for the critical ratio  $x = x_c = 4.05$ , indicating the fixed-point collision. (d) Same as (b), but for  $x = 3.7$ . The absence of the stable CBZ fixed point implies a direct continuous transition between a chiral-symmetry-broken state, signaled by the runaway flow towards  $G_2 \rightarrow +\infty$ , and a different ordered ground state, signaled by the runaway flow towards  $(G_1, G_2) \rightarrow (-\infty, -\infty)$ . Figure reprinted from Ref. [283].

points and the resulting slow flow on the XY side of the transition.

### 3.8. Quantum chromodynamics

As a final example for a continuous order-to-order transition, we discuss a model relevant to high-energy physics, namely quantum chromodynamics in  $3 + 1$  space-time dimensions, supplemented with local four-fermion interactions, defined by the action [217, 276, 292–



**Figure 3.15.:** Schematic phase diagram of many-flavor quantum chromodynamics in 3 + 1 space-time dimensions as a function of the ratio  $x = N_{\text{flavor}}/N_{\text{color}}$ , where  $N_{\text{flavor}}, N_{\text{color}} \rightarrow \infty$  (Veneziano limit). IR (UV) denotes the infrared (ultraviolet). The critical ratio  $x_c = 4.05$  is defined by the saddle-node bifurcation (Fig. 1.6(a)) between the Caswell-Banks-Zaks fixed point and the quantum critical point associated with chiral symmetry breaking, while the ratio  $x_{\text{CBZ}} = 11/2$  is defined by the transcritical bifurcation (Fig. 1.6(b)) between the Caswell-Banks-Zaks fixed point and the Gaussian fixed point. Below the conformal window,  $x < x_c$ , the Caswell-Banks-Zaks fixed point is no longer located in the physical space of real couplings, giving rise to the hadronic phase in which quarks are confined. At least in the Veneziano limit, the proposed fixed-point-annihilation mechanism suggests a continuous order-to-order quantum phase transition within the hadronic phase. Figure inspired by Ref. [294].

294, 420]

$$S_{\text{QCD}_4\text{-GN}} = \int d^4x \left[ i\bar{\psi} \not{D} \psi + \frac{1}{4} F_{\mu\nu}^z F_{\mu\nu}^z + \frac{1}{2} \sum_{\alpha=1}^4 G_{\alpha} \mathcal{O}_{\alpha} \right], \quad (3.54)$$

with covariant derivative  $\not{D} := \gamma_{\mu}(\partial_{\mu} - igA_{\mu})$ ,  $\text{SU}(N_{\text{color}})$  gauge field  $A_{\mu} := A_{\mu}^z T^z$ , and the associated field-strength tensor  $F_{\mu\nu} := F_{\mu\nu}^z T^z$ , where  $T^z$  denote the generators of  $\text{SU}(N_{\text{color}})$ . A Fierz-complete basis of local four-fermion interactions is given by [420]

$$\mathcal{O}_1 := (\bar{\psi}^a \gamma_{\mu} \psi^b)^2 + (\bar{\psi}^a \gamma_{\mu} \gamma_5 \psi^b)^2, \quad (3.55)$$

$$\mathcal{O}_2 := (\bar{\psi}^a \psi^b)^2 - (\bar{\psi}^a \gamma_5 \psi^b)^2, \quad (3.56)$$

$$\mathcal{O}_{3,4} := (\bar{\psi} \gamma_{\mu} \psi)^2 \pm (\bar{\psi} \gamma_{\mu} \gamma_5 \psi)^2. \quad (3.57)$$

For notational simplicity, we have abbreviated  $\bar{\psi}\psi \equiv \bar{\psi}_i^a \psi_i^a$  and  $(\bar{\psi}^a \psi^b)^2 \equiv \bar{\psi}_i^a \psi_i^b \bar{\psi}_j^b \psi_j^a$ , and assumed summation convention over repeated color indices  $i, j$  and flavor indices  $a, b$ . Consider the Veneziano limit  $N_{\text{flavor}}, N_{\text{color}} \rightarrow \infty$  with fixed ratio  $x := N_{\text{flavor}}/N_{\text{color}}$  [421]. In this limit, the flow equations reduce to a set of three equations which decouple from the flow of  $G_3$  and  $G_4$  [217, 293, 294],

$$\frac{dg^2}{d \ln b} = \frac{2(11-2x)}{3} g^4 + \frac{2(34-13x)}{3} g^6 - 2x g^4 G_1, \quad (3.58)$$

$$\frac{dG_1}{d \ln b} = -2G_1 - (1+x)G_1^2 - \frac{x}{4} G_2^2 + \frac{3}{4} g^4, \quad (3.59)$$

$$\frac{dG_2}{d \ln b} = -2G_2 + 2G_2^2 - 2xG_1G_2 + 6g^2G_2 + \frac{9}{2} g^4, \quad (3.60)$$

where we have rescaled the gauge coupling  $g^2 N_{\text{color}}/(16\pi^2) \mapsto g^2$  and the four-fermion couplings  $G_{1,2} \Lambda^2 N_{\text{color}}/(4\pi^2) \mapsto G_{1,2}$ . For reasons of consistency with the previous sections, we write the flow equations in terms of an renormalization group scale  $\ln b$ , with  $\ln b = 0$  ( $\ln b \rightarrow \infty$ ) corresponding to the ultraviolet (infrared) scale.

The fixed-point structure of the model in the Veneziano limit is shown schematically in

Fig. 3.14(a). For  $x > 11/2$ , asymptotic freedom is lost and the gauge coupling vanishes in the infrared if  $G_1$  and  $G_2$  are small at the ultraviolet scale. For  $x < 11/2$ , the gauge charge is a renormalization-group-relevant parameter and flows towards a finite value  $g^2 = g_\star^2(G_1) = (11 - 2x - 3xG_1)/(13x - 34)$  in the infrared. For  $x$  below, but not too far from  $11/2$ , the infrared attractive plane  $g^2 = g_\star^2(G_1)$  hosts a fully infrared stable fixed point that can be associated with the conformal phase of many-flavor quantum chromodynamics, the so-called Caswell-Banks-Zaks fixed point (CBZ) [422, 423]. In addition, the plane features two fixed points with unique infrared relevant directions, which can be associated to quantum critical points (QCP<sub>1</sub> and QCP<sub>2</sub>). For decreasing  $x$ , the infrared stable CBZ fixed point and one of the two quantum critical points (QCP<sub>2</sub>) approach each other and eventually collide at the critical value  $x_c = 4.05$ . This is illustrated in Figs. 3.14(b)–(d), which show the renormalization group flow in the plane  $g^2 = g_\star^2(G_1)$  for different values of  $x$ . The runaway flow towards  $G_2 \rightarrow +\infty$  at finite renormalization group scale is usually associated with an instability towards a chiral-symmetry-breaking ground state [272, 276, 292]. The other quantum critical point (QCP<sub>1</sub>) in the infrared attractive plane  $g^2 = g_\star^2(G_1)$ , however, is not affected by the fixed-point annihilation and persists for all  $x > 0$ . Since the runaway flow corresponding to this quantum critical fixed point is towards  $(G_1, G_2) \rightarrow (-\infty, -\infty)$ , one should expect it to describe an instability towards a different ground state, spontaneously breaking a symmetry different from chiral symmetry. As summarized in Fig. 3.15, this suggests a continuous order-to-order transition for  $x < x_c$  governed by QCP<sub>1</sub>, demonstrating the relevance of the proposed fixed-point-annihilation mechanism beyond the realm of condensed matter physics.

### 3.9. Conclusion

We have proposed a mechanism for generic continuous order-to-order phase transitions that operates independently of fractionalization. This mechanism is based on the collision and annihilation of two renormalization group fixed points, a quantum critical fixed point and an infrared stable fixed point associated with a disordered phase. A distinctive experimental or numerical signature of the proposed mechanism is a pronounced asymmetry in the effective energy scales between the two sides of the transition, characterized by a significantly lower critical temperature and a much smaller order parameter on the side where the order emerges from fixed-point annihilation. We have illustrated this mechanism in a Luttinger fermion system via a nonperturbative renormalization group approach, revealing a continuous antiferromagnetic-Weyl-to-nematic-insulator transition that is experimentally relevant to the pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ , where  $R$  denotes a rare-earth element. Additionally, the fixed-point-annihilation mechanism manifests in kagome quantum magnets and suggests a continuous quantum phase transition between a chiral spin liquid and valence bond solid. Apart from its experimental relevance, this quantum phase transition illustrates that the proposed mechanism can also occur in conjunction with fractionalization. Moreover, we have illustrated the mechanism in an anisotropic spin-boson model, in

consistency with recent results from exact quantum Monte Carlo simulations. Finally, we have argued that quantum chromodynamics with supplemental four-fermion interactions realizes a continuous order-to-order transition within the hadronic phase, demonstrating the relevance of the proposed fixed-point-annihilation mechanism in a high-energy setting.

The proposed and illustrated fixed-point-annihilation mechanism for continuous order-to-order transitions should be contrasted with previously discussed scenarios in which fixed-point annihilation gives rise to weakly first-order transitions, cf. Sec. 1.4.3. These scenarios, some of which relevant to  $SU(N)$  quantum spin models on two-dimensional lattices [153, 271], all rely on the collision and annihilation of a critical fixed point with a bicritical fixed point, thereby forbidding a continuous order-to-order transition after the annihilation without additional fine tuning. In contrast, all examples presented in this chapter feature a critical fixed point that collides and annihilates with an infrared stable fixed point, thereby destabilizing the associated interacting symmetric phase and giving rise to a generic continuous transition between two distinct orders.

The findings presented in this chapter suggest numerous promising directions for future research, some of which are outlined below. On the experimental side, careful investigations of rare-earth pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$  [303, 346] could reveal the proposed continuous phase transition between the AIAO antiferromagnetic Weyl semimetal and the nematic topological insulator. If the expected asymmetry in energy scales proves too strong, an indirect observation of the proposed quantum phase transition at intermediate temperatures might be pursued. This could involve identifying a transition from the AIAO antiferromagnetic Weyl semimetal phase to a paramagnetic phase exhibiting quantum critical behavior (with drifting critical exponents, cf. Chap. 2), followed by a crossover into the LAB regime (with drifting scaling exponents) as the  $R^{3+}$  ionic radius effectively increases through variations in chemical or hydrostatic pressure. Similar strategies might be employed to achieve a numerical observation, at zero or finite temperature, of the proposed fixed-point-annihilation mechanism and the resulting continuous order-to-order transition.

Furthermore, it would be interesting to study the QED<sub>3</sub>-Gross-Neveu model in a partially bosonized framework using a dynamical bosonization scheme. This approach would allow a direct observation of the putative continuous phase transition between the chiral spin liquid and the valence bond solid. Alternatively, one might integrate out the fermion fields completely [424], resulting in an order-parameter theory that could be compared to the nonlinear sigma model supplemented with a Wess-Zumino-Witten term. This procedure may further illuminate the similarities and differences between continuous order-to-order phase transitions from the proposed fixed-point-annihilation mechanism and the fractionalization mechanism, cf. Sec. 1.4. Moreover, the reduced spatial dimensionality may facilitate a direct numerical observation with the precision required to resolve the expected asymmetry in energy scales. Further, layered kagome materials such as herbertsmithite  $\text{ZnCu}_3(\text{OH})_6\text{Cl}_2$  [386], kapellasite  $\text{Cu}_3\text{Zn}(\text{OH})_6\text{Cl}_2$  [387], and  $\text{YCu}_3(\text{OH})_{6.5}\text{Br}_{2.5}$  [388, 389] provide an alternative promising platform for the experimental detection of a continuous order-to-order quantum phase transition.

Analogous to the  $\text{QED}_3$ -Gross-Neveu model, quantum chromodynamics with supplemental four-fermion interactions may in the future be investigated in a partially bosonized framework using a dynamical bosonization scheme. However, before doing so, one may first try to gain a better understanding, for instance via mean-field theory, of the annihilation-surviving quantum critical point  $\text{QCP}_1$  (Fig. 3.14), which is far less understood than  $\text{QCP}_2$ , the quantum critical point typically associated with chiral symmetry breaking. From a condensed-matter perspective, it would be interesting to investigate potential implications of our findings for quantum critical points involving symmetry-protected topological phases, as found in  $(3 + 1)$ -dimensional non-Abelian gauge theories [264].

Motivated by the identified continuous Ising-to-XY quantum phase transition in the anisotropic spin-boson model, one might attempt to find a continuous order-to-order *thermal* phase transition in future work, either within the context of spin-boson models<sup>3</sup> or in an unrelated setting. Although there is no fundamental reason why such realizations should be forbidden, a positive finding would still be interesting and would further set the proposed fixed-point-annihilation mechanism apart from the fractionalization mechanism for continuous order-to-order *quantum* phase transitions. As briefly mentioned in Sec. 3.2, the change in flow topology associated with the proposed mechanism suggests the existence of other exotic continuous phase transitions without fine tuning. This includes, for instance, continuous transitions between two distinct topologically ordered phases, or between a quasi-long-range-ordered phase and a long-range-ordered phase. One may try to identify and investigate concrete realizations in the future. Lastly, one could also attempt to construct other bifurcation-based mechanisms for novel criticality, drawing on other local or even global types of bifurcations in the renormalization group flow [217, 427, 428].

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<sup>3</sup>We note that an identification in the context of spin-boson models may be difficult, as certain quantum critical points lack a quantum-to-classical correspondence [289, 425, 426].

## 4. Critical point formalism for Luttinger fermion systems

*The most important skill in physics  
is the ability to neglect!* [429]

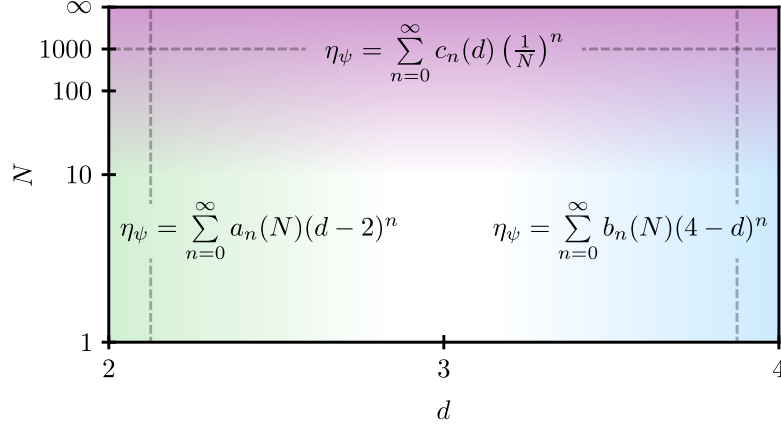
— attributed to Lev D. Landau

Whether one agrees with the above statement fully or only in a weakened form may be debated, but choosing a suitable approximation scheme, one that retains as much of the physics as possible while remaining feasible, is a crucial step in tackling any problem lacking an exact solution. In particular, this applies to interacting statistical and quantum field theories, which are usually not exactly soluble and have stimulated the development of numerous perturbative and nonperturbative methods. A particularly successful method for relativistic field theories is the large- $N$  expansion, in which physical quantities are computed perturbatively in the inverse number,  $1/N$ , of order-parameter components, fermion flavors, or gauge-group colors [95, 430]. Equivalent large- $N$  approaches have been attempted for nonrelativistic field theories, but results are usually more difficult to obtain due to the absence of Lorentz symmetry [431–434]. In this chapter, we introduce a novel large- $N$  framework by extending the relativistic critical point formalism [100–102], designed to compute the critical exponents of relativistic critical fixed points perturbatively in  $1/N$ , to nonrelativistic field theories with Fermi nodes. Specifically, we investigate the large- $N$  behavior of fermions close to  $N$  quadratic band touching points, coupled to an  $s$ -wave superconducting order parameter. This  $(3+1)$ -dimensional nonrelativistic Luttinger fermion system is of experimental relevance for the rare-earth pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ . In the course of this analysis, we also explore relativistic and random Luttinger fermion systems by means of critical point techniques.

This chapter is organized as follows. We begin by motivating the ongoing research presented in this chapter, see Sec. 4.1. Afterwards, we analyze both a nonrelativistic and a relativistic Luttinger fermion model at a large but finite number of fermion flavors  $N$ , see Secs. 4.2 and 4.3, respectively<sup>1</sup>. Finally, we give a brief outlook on a nonrelativistic Luttinger model with a random Yukawa interaction in a Veneziano-like limit, see Sec. 4.4, and conclude in Sec. 4.5.

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<sup>1</sup>Readers unfamiliar with the large- $N$  critical point formalism may find it helpful to read these two sections in reverse order instead of in the chronological order presented here.



**Figure 4.1.:** Critical exponents of Luttinger fermion systems with  $N = 1$  quadratic band touching point in  $d = 3$  spatial dimensions can be obtained perturbatively within a  $2 + \epsilon$  ( $4 - \epsilon$ ) expansion, as indicated by the series in  $d - 2$  ( $4 - d$ ) on the left (right) hand side for a representative critical exponent  $\eta_\psi$ . In general, the corresponding coefficients  $a_n(N)$  ( $b_n(N)$ ) depend on the number of quadratic band touching points  $N$ . In this chapter, we introduce a large- $N$  critical point formalism suitable for nonrelativistic systems with Fermi nodes. As displayed on the upper part, the resulting series in  $1/N$  for the representative critical exponent  $\eta_\psi$  involves coefficients  $c_n(d)$  that, in general, depend on the spatial dimension  $d$ . Marked by dashed lines, the regime of small  $d - 2$  ( $4 - d$ ) and large  $N$  allows for a cross-check between the  $2 + \epsilon$  ( $4 - \epsilon$ ) expansion and the large- $N$  formalism by expanding the coefficients in small  $1/N$  and  $\epsilon$ , respectively.

## 4.1. Motivation

This chapter involves several closely related research projects, which, however, pursue different goals. To account for this, we first present the formal motivation for developing a large- $N$  critical point formalism, and then turn to the more physics-oriented, conceptual motivation of higher-dimensional Sachdev-Ye-Kitaev (SYK) fixed points.

### 4.1.1. Formal motivation

Strongly coupled statistical and quantum field theories are inherently difficult to analyze, as they lack controlled expansions in small coupling constants. One way to address this problem is to organize perturbation theory in terms of an alternative small parameter. As discussed in Sec. 1.1.3, this is precisely the strategy underlying the  $\epsilon$  expansion: quantities such as critical or scaling exponents are computed perturbatively in the parameter  $\epsilon = d_c - d$ , allowing us to probe an originally  $d$ -dimensional theory right below its upper critical dimension  $d_c$  [70, 71]. Alternatively, one may consider an expansion in the inverse number,  $1/N$ , of order-parameter components, gauge-group colors, or fermion flavors [95, 430]. Seminal works include Stanley’s investigation of the  $O(N)$  transition [435], ’t Hooft’s analysis of quantum chromodynamics with  $U(N)$  gauge structure [436], and Gross and Neveu’s study of  $N$ -flavor Dirac fermions supplemented with local four-fermion interaction [437]. The critical properties of these relativistic theories<sup>2</sup> can be examined

<sup>2</sup>For clarity, we note that quantum chromodynamics is conveniently studied as a function of the inverse number of flavors, rather than colors (as might be inferred from ’t Hooft’s analysis), within the large- $N$  critical point formalism [438].

order by order within the large- $N$  critical point formalism [95, 100–102].

Systems without fundamental or emergent Lorentz symmetry, however, are generally more difficult to analyze, especially within a large- $N$  framework. This theme was initially explored in 2+1 dimensions in the context of fermions in the vicinity of an extended Fermi surface coupled to a  $U(1)$  gauge field [439–442]. However, the effective (one patch) theory is believed to remain strongly coupled in the limit of infinitely many flavors,  $N \rightarrow \infty$ , due to the abundance of gapless fermionic excitations [443]. While several nonrelativistic quantum field theories with a stable large- $N$  limit are nowadays known, see Secs. 2.3 and 3.4 for example, their  $1/N$  corrections remain largely unexplored well below the upper critical dimension. One goal of the first project presented in this chapter is to extend the aforementioned relativistic large- $N$  critical point formalism [95, 100–102] to a framework suitable for nonrelativistic quantum field theories involving Fermi nodes. Specifically, we study a Luttinger semimetal system consisting of fermions near  $N$  quadratic band touching points subject to an  $s$ -wave superconducting interaction, and aim to compute critical exponents as a function of the inverse number of Fermi nodes,  $1/N$ , as illustrated in Fig. 4.1.

As elaborated in detail above, see especially Secs. 2.1.2 and 2.6, Luttinger semimetals are not only of theoretical interest but also of significant experimental relevance for the pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ , where  $R$  denotes a rare-earth element [303, 311–313, 318, 346]. More precisely, a nonrelativistic large- $N$  critical point formalism is expected to enhance our understanding of various quantum critical phenomena in Luttinger systems for the physical case  $(d, N) = (3, 1)$ , cf. Fig. 4.1, and, in turn, improve our understanding of strongly interacting materials such as the rare-earth pyrochlore iridates.

#### 4.1.2. Conceptual motivation

The SYK model is a remarkable system, being exactly soluble, nearly conformally invariant, and maximally chaotic in the large- $N$  limit [67–69, 444, 445]. This unique combination of exotic properties makes the SYK model a promising playground for exploring various intriguing many-body phenomena, such as non-Fermi-liquid behavior, holographic duality, and fast scrambling [446, 447]. While its low dimensionality is part of the reason for its tremendous success, one ultimately seeks to identify higher-dimensional analogs of the  $(0+1)$ -dimensional SYK model, with respect to both condensed-matter and high-energy physics. However, constructing generalizations with strongly interacting, scale-invariant fixed points has proven to be a difficult task [448, 449]. This is reflected in the fact that most lattices of SYK quantum dots either turn out to be scale-dependent (interdot coupling via four-fermion interactions gives rise to local quantum criticality [449–451]) or weakly coupled (interdot coupling via single-particle-hopping operators eventually gives rise to Fermi-liquid behavior [449, 452, 453]) in the infrared.

By considering a novel way of generalization, Kim et al. were able to identify SYK fixed points in  $D = 1 + 1$  and  $D = 2 + 1$  dimensions [449]. Their construction involves a random

variant of the partially bosonized Gross-Neveu model [437, 449],

$$S_{\text{GN-SYK}} = \int d^D x \left( \sum_{i=1}^N \bar{\psi}_i \gamma_\mu \partial_\mu \psi_i + \sum_{k=1}^M \frac{r}{2} \phi_k^2 + \sum_{i,j=1}^N \sum_{k=1}^M g_{ijk} \bar{\psi}_i \psi_j \phi_k \right), \quad (4.1)$$

where  $\psi_i$  and  $\bar{\psi}_i := \psi_i^\dagger \gamma_0$  denote  $N$  flavors of  $d_\gamma$ -component Dirac spinors, summation over repeated space-time indices  $\mu = 0, \dots, D-1$  is assumed, the  $d_\gamma \times d_\gamma$  matrices  $\gamma_\mu$  satisfy the Clifford algebra, and  $\phi_k$  denotes a real bosonic field with  $M$  components and mass  $r$ . The Yukawa couplings  $g_{ijk}$  are drawn from a complex, zero-mean Gaussian distribution with pseudo-variance  $\langle\langle g_{ijk} g_{lmn} \rangle\rangle = g^2 / N^2 \delta_{im} \delta_{jl} \delta_{kn}$  and  $g^2 \in \mathbb{R}$ . In the Veneziano-like limit [421], i.e., for  $M, N \rightarrow \infty$  with a fixed ratio of bosonic to total fermionic components  $\rho := M/(Nd_\gamma)$ , the above theory admits a nontrivial scale-invariant solution with interacting fermion and boson anomalous dimensions<sup>3</sup>,  $\eta_\psi \equiv \eta_\psi(\rho)$  and  $\eta_\phi \equiv \eta_\phi(\rho)$ , for all ratios  $0 < \rho < \infty$  [449]. In  $D = 1+1$  ( $D = 2+1$ ) dimensions, this solution corresponds to an infrared stable (quantum critical) fixed point. Analogous to the  $(0+1)$ -dimensional SYK model, the  $(1+1)$ -dimensional generalization<sup>4</sup> is maximally chaotic, i.e., its out-of-time-order correlators are characterized by the maximally allowed Lyapunov exponent  $\lambda = 2\pi T$ , where  $T$  denotes the temperature [449]. On a technical level, Kim et al. employ the relativistic large- $N$  critical point formalism [95, 100–102] to obtain the nontrivial scale-invariant solutions. Apart from intriguing chaotic behavior, random variants of Gross-Neveu models have recently been shown to display superconductivity around the quantum critical point [454, 455]. The combination of a superconducting dome and non-Fermi-liquid behavior makes these theories promising toy models for advancing our understanding of unconventional superconductors such as the cuprates.

The main goal of a project briefly discussed at the end of this chapter is to investigate the random variant of a partially bosonized Luttinger fermion model in  $3+1$  dimensions. Since Luttinger systems are known to share many similarities with Gross-Neveu field theories, such a model may host a similar SYK-like quantum critical fixed point in three spatial dimensions. If existent, follow-up research directions are manifold and may, for instance, include the investigation of out-of-time-order correlations [449] or superconductivity [454, 455]. To our knowledge, a quantum field theory in three spatial dimensions admitting such a nontrivial scale-invariant solution has not yet been identified. An investigation, however, first requires the existence of a nonrelativistic large- $N$  critical point formalism, cf. Sec. 4.1.1.

<sup>3</sup>The fermion scaling dimension  $\Delta = \Delta(\rho)$  computed in Ref. [449] is related to the fermion and boson anomalous dimensions via  $\eta_\psi = -D + 1 + 2\Delta$  and  $\eta_\phi = D + 2 - 4\Delta$ , respectively.

<sup>4</sup>We are not aware of any results on out-of-time-order correlators of the  $(2+1)$ -dimensional Gross-Neveu-SYK model.

## 4.2. Nonrelativistic Luttinger fermions at large $N$

In this first of three sections presenting ongoing research, we analyze a Luttinger fermion system at a large but finite number of fermion flavors  $N$ , building on the motivation given in Sec. 4.1.1.

### 4.2.1. Model

In principle, one could try to establish the nonrelativistic large- $N$  critical point formalism using (parts of) the models already encountered in Secs. 2.2 and 3.3.2. However, both all-in-all-out and nematic order seem not to be ideal choices here, since the former requires the inclusion of spatial anisotropy  $\delta$ , which is an additional complication, and the latter constitutes a five-component tensor order-parameter field, which is a comparatively complicated degree of freedom. To keep the analysis as simple as possible, we choose to work with  $N$  flavors of Luttinger fermions coupled to an  $s$ -wave superconducting order parameter<sup>5</sup> in this section. As before, noninteracting Luttinger fermions with  $\text{SO}(3)$  rotation symmetry,  $\delta = 0$ , and particle-hole symmetry are characterized by the Euclidean action [281, 328–333, 457]

$$S_0 = \int d\tau d^3x \sum_{i=1}^N \psi_i^\dagger \left[ \partial_\tau + \sum_{a=1}^5 d_a (-i\nabla) \gamma_a \right] \psi_i, \quad (4.2)$$

where  $\psi_i$  denotes a four-component Grassmann field and the  $4 \times 4$  matrices  $\gamma_a$  fulfill the Euclidean Clifford algebra,  $\{\gamma_a, \gamma_b\} = 2\delta_{ab} \mathbb{1}$ . Further, the family of functions  $d_a$  are proportional to the  $\ell = 2$  real spherical harmonics, viz.  $d_1(\vec{p}) = \sqrt{3}p_y p_z$ ,  $d_2(\vec{p}) = \sqrt{3}p_x p_z$ ,  $d_3(\vec{p}) = \sqrt{3}p_x p_y$ ,  $d_4(\vec{p}) = \frac{\sqrt{3}}{2}(p_x^2 - p_y^2)$ , and  $d_5(\vec{p}) = \frac{1}{2}(2p_z^2 - p_x^2 - p_y^2)$ . For  $N = 1$ , we recover the experimentally most relevant case of a single quadratic band touching point. The above theory is coupled to a complex scalar field  $\phi$  via an  $s$ -wave interaction governed by the action [327, 458, 459]

$$S_\phi = \int d\tau d^3x \left[ r|\phi|^2 + g \sum_{i=1}^N \left( \phi \psi_i^\dagger \gamma_{45} \psi_i^* + \phi^* \psi_i^\dagger \gamma_{45} \psi_i \right) \right], \quad (4.3)$$

where  $r$  denotes the order-parameter mass,  $g$  is the Yukawa coupling, and, as before,  $\gamma_{45} := i\gamma_4\gamma_5$  is constructed from the two imaginary Euclidean Dirac matrices. Superconductivity ( $s$ -wave) is indicated by a finite expectation value,  $\langle \phi \rangle \neq 0$ , and opens a rotationally invariant gap in the electronic spectrum [458, 460]. In the following, we aim to analyze the above theory  $S = S_0 + S_\phi$  for a large but finite number of band touching points  $N$  and in general spatial dimension  $2 < d < 4$ .

<sup>5</sup>The choice of a superconducting order parameter comes with the additional advantage that certain diagrammatic contributions are absent due to charge conservation. In particular, this applies to two-loop contributions (see Fig. 1 of Ref. [456], for instance), which, at least in the context of Gross-Neveu theories, are known to contribute to the critical exponents already at order  $\mathcal{O}(1/N)$  [95].

### 4.2.2. Towards a critical solution

#### Ansatz

As in the relativistic large- $N$  critical point formalism, we begin our analysis by writing down an ansatz for the critical ( $r = 0$ ), interacting propagators  $\tilde{G}_\psi$  and  $\tilde{G}_\phi$  in frequency and momentum space

$$\tilde{G}_\psi(\omega, \vec{k}) = \frac{A}{k^{2-\eta_\psi}} \left[ i \operatorname{sgn}(\omega) \tilde{g}_\psi^{(1)} \left( \frac{|\omega|}{k^z} \right) \mathbb{1} + \tilde{g}_\psi^{(2)} \left( \frac{|\omega|}{k^z} \right) d_a(\hat{k}) \gamma_a \right] + \dots, \quad (4.4)$$

$$\tilde{G}_\phi(\omega, \vec{k}) = \frac{B}{k^{2-\eta_\phi}} \tilde{g}_\phi \left( \frac{|\omega|}{k^z} \right) + \dots, \quad (4.5)$$

where  $A$  and  $B$  denote frequency- and momentum-independent amplitudes. For better readability, we keep the flavor-diagonal structure of the fermion propagator implicit in this section. Further,  $\eta_\psi$  and  $\eta_\phi$  represent the fermion and boson anomalous dimensions, respectively. Owing to the nonrelativistic nature of the problem, the dynamical exponent  $z$  and three dimensionless real-valued scaling functions,  $\tilde{g}_\psi^{(1)}$ ,  $\tilde{g}_\psi^{(2)}$ , and  $\tilde{g}_\phi$ , have to be incorporated. In this section, summation over repeated indices  $a = 1, \dots, 5$  is assumed and  $\hat{k} := \vec{k}/k$ , with  $k := |\vec{k}|$ , such that  $d_a(\hat{k})$  become the  $\ell = 2$  real spherical harmonics<sup>6</sup>,  $d_a(\hat{k}) = \sqrt{\frac{d}{2(d-1)}} \hat{k}^\top \Lambda_a \hat{k}$ , where  $\Lambda_a$  denote the real Gell-Mann matrices [331]. The ellipses denote corrections-to-scaling terms that, at order  $1/N$ , are expected to contribute to the boson anomalous dimension  $\eta_\phi$  and the correlation-length exponent  $\nu$ , but not to the fermion anomalous dimension  $\eta_\psi$ , which we aim to compute in the following. At this point in the analysis, it is unclear whether corrections-to-scaling terms contribute to the dynamical  $z$  at order  $1/N$  or not.

Note that the above ansatz is not the most general one could write down, since we have assumed a unique dynamical exponent  $z$  and a unique fermion anomalous dimension  $\eta_\psi$ . Despite these (reasonable) assumptions and neglecting corrections-to-scaling contributions for now, the ansatz involves three interacting scaling functions and two different tensor structures in the fermionic sector, making it far more difficult to handle than any leading-order relativistic ansatz. In order to use a similar strategy to the relativistic large- $N$  critical point formalism, namely, solving the Dyson-Schwinger equations for the two-point functions in real space, we employ a simplified asymptotic ansatz

$$\tilde{G}_\psi(\omega, \vec{k}) \simeq \frac{A}{k^{2-\eta_\psi}} \left| \frac{\omega}{k^z} \right|^{\Delta_\psi} d_a(\hat{k}) \gamma_a, \quad (4.6)$$

$$\tilde{G}_\phi(\omega, \vec{k}) \simeq \frac{B}{k^{2-\eta_\phi}} \left| \frac{\omega}{k^z} \right|^{\Delta_\phi}, \quad (4.7)$$

with the expectation that it reproduces at least certain features of the full solution. In the regime  $|\omega| \ll k^z$ , the fermionic contribution with trivial tensor structure, the Berry-phase term, appears to be subleading and the leading-order exponents  $\Delta_\psi, \Delta_\phi \rightarrow 0$  for  $N \rightarrow \infty$ .

<sup>6</sup>In the previous chapter, the  $\ell = 2$  real spherical harmonics are denoted by  $\tilde{d}_a$ . In order to keep the notation as overseeable as possible, we use  $d_a(\hat{k}) \equiv \tilde{d}_a(\theta, \phi)$  here instead.

### Fourier transform

Following the relativistic large- $N$  critical point formalism, we now take the inverse Fourier transform  $\mathcal{F}^{-1}[\cdot] = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\vec{x}} \int \frac{d\omega}{2\pi} e^{i\omega\tau}$ , with respect to both frequency and momentum, of the above asymptotic propagators as well as their inverses, which yields four functions denoted  $G_\psi$  and  $G_\phi$  as well as  $G_\psi^{-1}$  and  $G_\phi^{-1}$ , respectively<sup>7</sup>. Using the master formulae provided in App. B.1, we arrive at the expressions

$$G_\psi(\tau, \vec{x}) \simeq A \frac{2^{-2+\eta_\psi+(1-z)\Delta_\psi} (2 - \eta_\psi + z\Delta_\psi - d) a_d \left( \frac{2-\eta_\psi+z\Delta_\psi}{2} \right) a_1 \left( -\frac{\Delta_\psi}{2} \right)}{\pi^{\frac{d+1}{2}} (2 - \eta_\psi + z\Delta_\psi)} \times \frac{1}{x^{d+z-2+\eta_\psi}} \left| \frac{\tau}{x^z} \right|^{-1-\Delta_\psi} d_a(\hat{x}) \gamma_a, \quad (4.8)$$

$$G_\psi^{-1}(\tau, \vec{x}) \simeq \frac{1}{A} \frac{2^{2-\eta_\psi-(1-z)\Delta_\psi} (-2 + \eta_\psi - z\Delta_\psi - d) a_d \left( \frac{-2+\eta_\psi-z\Delta_\psi}{2} \right) a_1 \left( \frac{\Delta_\psi}{2} \right)}{\pi^{\frac{d+1}{2}} (-2 + \eta_\psi - z\Delta_\psi)} \times \frac{1}{x^{d+z+2-\eta_\psi}} \left| \frac{\tau}{x^z} \right|^{-1+\Delta_\psi} d_a(\hat{x}) \gamma_a, \quad (4.9)$$

$$G_\phi(\tau, \vec{x}) \simeq B \frac{2^{-2+\eta_\phi+(1-z)\Delta_\phi} a_d \left( \frac{2-\eta_\phi+z\Delta_\phi}{2} \right) a_1 \left( -\frac{\Delta_\phi}{2} \right)}{\pi^{\frac{d+1}{2}}} \frac{1}{x^{d+z-2+\eta_\phi}} \left| \frac{\tau}{x^z} \right|^{-1-\Delta_\phi}, \quad (4.10)$$

$$G_\phi^{-1}(\tau, \vec{x}) \simeq \frac{1}{B} \frac{2^{2-\eta_\phi-(1-z)\Delta_\phi} a_d \left( \frac{-2+\eta_\phi-z\Delta_\phi}{2} \right) a_1 \left( \frac{\Delta_\phi}{2} \right)}{\pi^{\frac{d+1}{2}}} \frac{1}{x^{d+z+2-\eta_\phi}} \left| \frac{\tau}{x^z} \right|^{-1+\Delta_\phi}, \quad (4.11)$$

where  $a_n(\alpha) := \Gamma(n/2 - \alpha)/\Gamma(\alpha)$  with  $n = 1, d$ .

### Skeleton Dyson-Schwinger equations

In the next step, we insert the above asymptotic expressions into the Dyson-Schwinger equations for the two-point functions in real space and imaginary time [95, 461, 462]

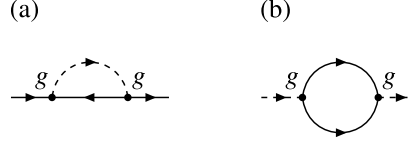
$$G_\psi^{-1}(\tau, \vec{x}) = G_{\psi,0}^{-1}(\tau, \vec{x}) + \Sigma_\psi(\tau, \vec{x}), \quad (4.12)$$

$$G_\phi^{-1}(\tau, \vec{x}) = G_{\phi,0}^{-1}(\tau, \vec{x}) + \Sigma_\phi(\tau, \vec{x}), \quad (4.13)$$

where  $G_{\psi,0}^{-1}$  ( $G_{\phi,0}^{-1}$ ) denotes the inverse Fourier transform of the inverse bare fermion (boson) propagator and, analogously,  $\Sigma_\psi$  ( $\Sigma_\phi$ ) represents the inverse Fourier transform of the fermion (boson) self-energy<sup>8</sup>. At  $N < \infty$ , both contributions from the bare propagators,  $G_{\psi,0}^{-1}$  and  $G_{\phi,0}^{-1}$ , are subleading compared to the contributions from the simplified interacting propagators,  $G_\psi^{-1}$  and  $G_\phi^{-1}$ , and can therefore be omitted [95]. Interaction effects are taken

<sup>7</sup>Following the standard notation in the large- $N$  critical point formalism community [95], we emphasize that  $G_\psi^{-1}$  and  $G_\phi^{-1}$  are not the inverses of  $G_\psi$  and  $G_\phi$ , respectively, but are the (inverse) Fourier transforms of the  $\tilde{G}_\psi^{-1}$  and  $\tilde{G}_\phi^{-1}$ , which indeed are the inverses of  $\tilde{G}_\psi$  and  $\tilde{G}_\phi$  in frequency and momentum space.

<sup>8</sup>Note that by transforming the one-loop skeleton Dyson-Schwinger equations from frequency and momentum space to real space and imaginary time, we trade two coupled integral equations for two coupled algebraic equations. The latter is, at least in the relativistic case, a much better starting point, since the inverse Fourier transforms of propagators and their inverses are readily obtained, cf. Sec. 4.3.



**Figure 4.2.:** Skeleton diagrams at the one-loop order contributing to the (a) fermion and (b) boson self-energies. While solid (dashed) lines denote dressed fermion (complex boson) propagators, dots represent bare Yukawa vertices.

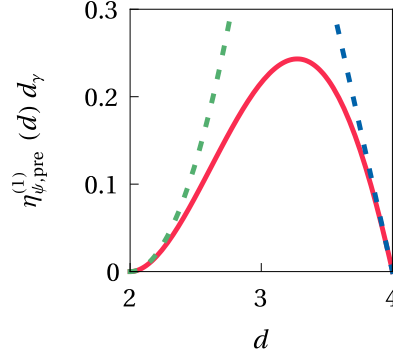
into account by means of a skeleton expansion of the self-energies, i.e., by a resummed loop expansion using dressed propagators and bare Yukawa vertices [78, 95, 463–465]. To leading order, the self-energies contribute through the resummed one-loop diagrams displayed in Fig. 4.2.

Inserting the asymptotic expressions Eqs. (4.8)–(4.11) into the one-loop skeleton Dyson-Schwinger equations results in two rather lengthy equations in several variables. As expected from the relativistic large- $N$  critical point formalism, both equations contain the product  $A^2Bg^2$  once. Although it is tempting to eliminate this factor and arrive at a single equation, we first want to have a look at both equations separately. In agreement with the relativistic large- $N$  critical point formalism, the spatial contributions in both equation cancel in the limit  $N \rightarrow \infty$ , which is a direct consequence of correctly incorporating the effective dimensionality  $d + z$  in Eqs. (4.8)–(4.11). As a byproduct, this confirms both anomalous dimensions,  $\eta_\psi \simeq \eta_\psi^{(0)}(d) = 0$  and  $\eta_\phi \simeq \eta_\phi^{(0)}(d) = 4 - d$ , as known from random-phase approximation. This behavior contrasts with the temporal contributions in both equations, or more precisely powers of  $|\tau/x^z|$ , which do not cancel as  $N \rightarrow \infty$ . According to our current knowledge, this is an artifact which, most likely, should be attributed to the simplified asymptotic ansatz in frequency and momentum space, Eqs. (4.6)–(4.7), that can apparently only reproduce certain features but not the full solution. On a related note, both equations contain factors of  $a_1(\pm\Delta_i/2)$ , where  $i = \psi, \phi$ , whose large- $N$  behavior is unclear at this point and is subject of ongoing research. From a formal perspective, they arise from the inverse Fourier transforms with respect to frequency. One may hypothesize that their behavior results from the expansion  $\Delta_i \simeq 0 + \Delta_i^{(1)}(d)/N$ , which characterizes the asymptotic behavior of the corresponding interacting scaling function in frequency and momentum space. In contrast, the inverse Fourier transforms with respect to momentum seem to give more reliable contributions. This becomes apparent, for instance, from the fermionic contribution  $G_\psi^{-1}$ , Eq. (4.9), which vanishes in the limit  $N \rightarrow \infty$ . Although this feature may seem like an unimportant detail, it is a crucial ingredient for a successful large- $N$  expansion, since it balances with the fermion self-energy  $\Sigma_\psi$ , cf. Eq. (4.12). In other words, the above Dyson-Schwinger equations, Eqs. (4.12)–(4.13), take the form

$$\mathcal{O}(1/N) = A^2Bg^2 \times \mathcal{O}(1), \quad (4.14)$$

$$\mathcal{O}(1) = A^2Bg^2 \times N \times \mathcal{O}(1), \quad (4.15)$$

with  $A^2Bg^2 = \mathcal{O}(1/N)$  and neglecting the potentially ill-defined contributions from the frequency Fourier transform for now. Of course, discarding these terms is not a valid



**Figure 4.3.:** Preliminary first-order contribution  $\eta_{\psi,\text{pre}}^{(1)}(d) d_\gamma$  (solid line) to the fermion anomalous dimension  $\eta_\psi = \sum_{n=0}^{\infty} \eta_\psi^{(n)}(d) (1/N)^n$  obtained within the nonrelativistic large- $N$  framework. In  $2 < d < 4$  spatial dimensions,  $\eta_{\psi,\text{pre}}^{(1)}(d) > 0$  results from a preliminary, partial solution of the Dyson-Schwinger equation and characterizes an interacting quantum critical fixed point to  $s$ -wave superconducting order. The dashed green (blue) line visualizes the quadratic (linear) asymptotic behavior near the lower (upper) critical dimension.

step of calculation, however, at the current level, it may be helpful in gaining a better understanding of the ansatz, which may eventually yield the full solution of the Dyson-Schwinger equations. On this basis, we finally eliminate the factor  $A^2 B g^2$  and arrive at a single equation for the first-order contribution

$$\eta_{\psi,\text{pre}}^{(1)}(d) = \frac{-16\Gamma(d-1)}{(d-2)(d+2)\Gamma\left(\frac{2-d}{2}\right)\Gamma^2\left(\frac{d-2}{2}\right)\Gamma\left(\frac{d+2}{2}\right)d_\gamma} \quad (4.16)$$

to the fermion anomalous dimension  $\eta_\psi = \sum_{n=0}^{\infty} \eta_\psi^{(n)}(d) (1/N)^n$ , cf. Fig. 4.1, where we have added a subscript to highlight that this is not the final result but a preliminary expression. For clarity, we have kept the number of spinor components  $d_\gamma = 4$  general above. As visualized in Fig. 4.3, the preliminary first-order contribution  $\eta_{\psi,\text{pre}}^{(1)}(d)$  displays multiple features expected from a well-defined large- $N$  solution. This includes linear behavior

$$\eta_{\psi,\text{pre}}^{(1)}(d) = \frac{2(4-d)}{3d_\gamma} + \mathcal{O}\left((4-d)^2\right) \quad (4.17)$$

in the vicinity of the upper critical dimension, quadratic behavior

$$\eta_{\psi,\text{pre}}^{(1)}(d) = \frac{(d-2)^2}{2d_\gamma} + \mathcal{O}\left((d-2)^3\right) \quad (4.18)$$

near the lower critical dimension, and  $\eta_{\psi,\text{pre}}^{(1)}(d) > 0$  for all spatial dimensions  $2 < d < 4$ . These promising features are known, for instance, from Gross-Neveu theories [95] and will be encountered again in the subsequent section, Sec. 4.3, in the context of relativistic Luttinger fermion theories. Apart from this intuitive comparison, Eq. (4.17) allows for a quantitative cross-check with the first-order contribution

$$\tilde{\eta}_\psi^{(1)}(N) = \frac{8}{8+9Nd_\gamma} = \frac{8}{9Nd_\gamma} + \mathcal{O}\left(\frac{1}{N^2}\right) \quad (4.19)$$

to the fermion anomalous dimension  $\eta_\psi = \sum_{n=0}^{\infty} \tilde{\eta}_\psi^{(n)}(N)(4-d)^n$ , determined within a  $4 - \epsilon$  expansion for general  $N$  (not presented here). The above expression generalizes the result for a single ( $N = 1$ ) quadratic band touching point, as presented in Ref. [458]. An expansion of  $\tilde{\eta}_\psi^{(1)}(N)$  in  $1/N$  reveals a difference by a factor of  $4/3$  relative to the expanded large- $N$  result, Eq. (4.17). In the future, one may try to compute the fermion anomalous dimension  $\eta_\psi$  within a two-loop  $2 + \epsilon$  expansion. A successful calculation would allow for a comparison with the corresponding expanded large- $N$  result displayed in Eq. (4.18). In case this future cross-check also reveals a difference of  $4/3$ , one could speculate that the preliminary large- $N$  expression  $\eta_{\psi,\text{pre}}^{(1)}(d)$ , Eq. (4.16), only misses an overall factor of  $4/3$ . However, if this is not the case, it is clear that  $\eta_{\psi,\text{pre}}^{(1)}(d)$  misses a dimension-dependent contribution. We spare the reader further technical details and finish this section with a few, more general thoughts.

## Reflections

As shown above, we are currently able to reproduce several crucial features of the exact first-order contribution  $\eta_\psi^{(1)}(d)$ , but we still struggle to treat and understand some terms in the corresponding Dyson-Schwinger equations. One possibility is that these terms cause a difficult interplay of  $1/N$  corrections associated with the critical exponents, such as the fermion anomalous dimension  $\eta_\psi$  and the dynamical exponent  $z$ , and  $1/N$  corrections associated with the exponents  $\Delta_\psi$  and  $\Delta_\phi$ , the remnants of the interacting scaling functions. Alternatively, the problematic terms in the Dyson-Schwinger equations may in fact be ill-defined and just result from using an oversimplified asymptotic ansatz, Eqs. (4.6)–(4.7). Additionally, the Berry-phase contribution in the fermionic sector may not be negligible and complicate the analysis further, ultimately returning us to the original ansatz, Eqs. (4.4)–(4.5). As previously mentioned, even the original ansatz may be oversimplified and require the inclusion of multiple non-unique dynamical exponents and/or fermion anomalous dimensions, as known from other nonrelativistic quantum field theories [46]. We note that there are plenty of other, arguably more exotic, scenarios, including the influence of evanescent operators [466], which cannot be excluded at this point.

In order to avoid getting lost in mathematical details, we refrain from discussing the many other large- $N$  ansatzes, approaches, and reformulations that we have explored in this project, which ultimately led to difficulties similar to those described above. However, to give the reader a feeling for the complexity of the problem, we note that we have spent  $\mathcal{O}(100)$  more time constructing a large- $N$  solution for nonrelativistic Luttinger fermions than for relativistic Luttinger fermions, cf. Sec. 4.3. Similar challenges were faced in the 1970s by Vasil’ev, Pis’mak, and Honkonen, the pioneers of the relativistic large- $N$  critical point formalism, as reflected in the opening line of their seminal work: “The practical calculation of the critical exponents in the framework of the  $1/n$  expansion is much more difficult than analogous calculations in a  $4 - \epsilon$  expansion.” [101].

How to proceed? During the writing process of this dissertation, the author became aware of Refs. [431, 432], that report on a nonrelativistic large- $N$  critical point formalism suitable for scalar bosonic field theories. In particular, this framework can be used to investigate the multicritical behavior of  $m$ -axial Lifshitz points in  $d$  spatial dimensions, with  $0 \leq m \leq d$  [467, 468]. Choosing  $m = 1$  (or  $m = d - 1$ ), yields a field theory similar, though simpler, to our Luttinger fermion theory, since both theories possess a spatial/space-time anisotropy with exactly one exceptional direction, which results in a nontrivial anisotropy/dynamical exponent. We currently investigate how to further generalize the nonrelativistic large- $N$  critical point formalism presented in Refs. [431, 432], such that it can also account for nonrelativistic fermionic quantum field theories, like the Luttinger fermion system discussed in this section. Since the nonrelativistic large- $N$  critical point formalism introduced in Refs. [431, 432] operates entirely in momentum space, we investigate in parallel what can be learned from it for the real-space-imaginary-time approach presented in this section.

### 4.3. Relativistic Luttinger fermions at large $N$

In this second of three sections on ongoing research, we analyze a relativistic Luttinger fermion system with a large but finite number of fermion flavors  $N$ . The motivation for this is twofold. First, given the difficulties encountered in the previous section, we aim to build some intuition for this related, but relativistic, model. In this way, we can employ the well-established relativistic large- $N$  critical point formalism and be certain that any potential difficulties arise from the model and/or the ansatz, rather than from the method itself. Second, aside from being a system that could help to gain a better understanding of the large- $N$  behavior of nonrelativistic Luttinger semimetals, relativistic Luttinger fermion theories in  $D = 4$  space-time dimensions have been shown to be interesting in their own right and exhibit many unconventional features, including the realization of asymptotic freedom already in the fermionic matter sector [336–338]. In this section, we aim to gain a better understanding of these theories also in different space-time dimensions, such as  $D = 3$  or  $D = 5$ .

#### 4.3.1. Model

We begin our analysis with the definition of an interacting model of many-flavor relativistic Luttinger fermions. Specifically, we consider the partially bosonized Luttinger-Gross-Neveu model in  $D$ -dimensional Minkowski space-time with signature  $(+, -, -, \dots)$ , which is characterized by the action [336–338]

$$S_{\text{LGN}} = \int d^D x \left[ \sum_{i=1}^N \bar{\psi}_i (-G_{\mu\nu} \partial^\mu \partial^\nu + g\phi) \psi_i - \frac{r}{2} \phi^2 \right], \quad (4.20)$$

where  $\psi_i$  and  $\bar{\psi}_i$  denote  $N$  flavors of  $d_\gamma$ -component spinors<sup>9</sup>, summation over repeated indices  $\mu, \nu = 0, \dots, D-1$  is assumed, and  $G_{\mu\nu}$  represent Lorentz-traceless,  $d_\gamma \times d_\gamma$  matrices which satisfy the Lorentzian Abrikosov algebra<sup>10</sup> and are labeled by space-time indices. Further,  $g$  denotes the Yukawa coupling and  $r$  the order-parameter mass of the real scalar field  $\phi$ . In contrast to the ordinary Gross-Neveu model [437], the above Luttinger-Gross-Neveu model is Fierz-incomplete and generates an additional, tensorial channel  $\propto (\bar{\psi}_i G_{\mu\nu} \psi_i)^2$  under renormalization [336], which is not analyzed in this thesis. However, note that both channels decouple in the large- $N$  limit [336].

### 4.3.2. Scale-invariant solution

#### Ansatz

We now employ the relativistic large- $N$  critical point formalism [95, 100–102] to analyze the Luttinger-Gross-Neveu model  $S_{\text{LGN}}$ , Eq. (4.20). As a first step, we make an ansatz for the scale-invariant ( $r = 0$ ), interacting propagators  $\tilde{G}_\psi$  and  $\tilde{G}_\phi$  in momentum space

$$\tilde{G}_\psi(k) = A|k|^{-2+\eta_\psi} G_{\mu\nu} \hat{k}^\mu \hat{k}^\nu \left(1 + A'|k|^{-\lambda} + \dots\right), \quad (4.21)$$

$$\tilde{G}_\phi(k) = B|k|^{-2+\eta_\phi} \left(1 + B'|k|^{-\lambda} + \dots\right), \quad (4.22)$$

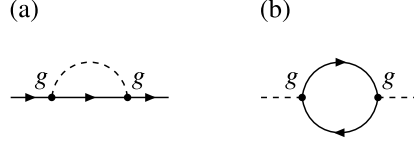
where  $A$ ,  $A'$ ,  $B$ , and  $B'$  denote momentum-independent amplitudes. Again, we keep the flavor-diagonal structure of the fermion propagator implicit in this section for better readability. Further, the nontrivial tensor structure  $G_{\mu\nu}$  is inherited from the Luttinger-Gross-Neveu action  $S_{\text{LGN}}$ ,  $\hat{k} := k/|k|$  (note the slight change in notation relative to the previous section), and the ellipses denote terms beyond the leading corrections-to-scaling contribution. The final ingredients of our ansatz are the fermion anomalous dimension  $\eta_\psi$ , the boson anomalous dimension  $\eta_\phi$ , and the exponent  $\lambda$ . The latter determines the correlation-length exponent  $\nu$  in the case of a critical fixed point,  $\lambda = 1/\nu$ , and is expected to coincide with the corrections-to-scaling exponent  $\omega$  (as encountered in renormalization group calculations, cf. Sec. 3.5.3) in the case of an infrared stable fixed point,  $\lambda = \omega$ .

#### Fourier transform

In a second step, we take the inverse Fourier transform  $\mathcal{F}^{-1}[\cdot] = \int \frac{d^D k}{(2\pi)^D} e^{ik_\mu x^\mu}$  of the propagators as well as their inverses, which yields four functions denoted  $G_\psi$  and  $G_\phi$ , as

<sup>9</sup>The reader is referred to Ref. [336] for an explicit construction of the conjugate spinor  $\bar{\psi}_i$  in the spin-base formalism. In  $D = 4$ , one requires  $d_\gamma = 32$  complex components, hence the large- $N$  expansion is an excellent choice here, as it is in fact an expansion in the inverse number of total fermionic components  $1/(Nd_\gamma) \ll 1$ , which is relatively small even for  $N = 1$  fermion flavor. In  $D = 3$  ( $D = 5$ ) space-time dimensions,  $d_\gamma = 4$  ( $d_\gamma = 128$ ) suffices [469].

<sup>10</sup>We refer the reader to Ref. [336] for the construction of the Lorentzian Abrikosov algebra, the counterpart of the Euclidean Abrikosov algebra given in Refs. [329, 331].



**Figure 4.4.:** Skeleton diagrams at the one-loop order contributing to the (a) fermion and (b) boson self-energies. While solid (dashed) lines denote dressed fermion (real boson) propagators, dots represent bare Yukawa vertices.

well as  $G_\psi^{-1}$  and  $G_\phi^{-1}$ , respectively. Using the master formulae provided in App. B.2, these functions take the leading-order asymptotic form

$$G_\psi(x) \simeq -A i^{1-D} 2^{-4+\eta_\psi} \pi^{-\frac{D}{2}} (-D+4-\eta_\psi) (D-2+\eta_\psi) \times a_D \left( \frac{4-\eta_\psi}{2} \right) G_{\mu\nu} \hat{x}^\mu \hat{x}^\nu |x|^{-D+2-\eta_\psi}, \quad (4.23)$$

$$G_\psi^{-1}(x) \simeq -A^{-1} i^{1-D} 2^{-\eta_\psi} \pi^{-\frac{D}{2}} (-D+\eta_\psi) (D+2-\eta_\psi) \times a_D \left( \frac{\eta_\psi}{2} \right) G_{\mu\nu} \hat{x}^\mu \hat{x}^\nu |x|^{-D-2+\eta_\psi}, \quad (4.24)$$

$$G_\phi(x) \simeq -B i^{1-D} 2^{-2+\eta_\phi} \pi^{-\frac{D}{2}} a_D \left( \frac{2-\eta_\phi}{2} \right) |x|^{-D+2-\eta_\phi}, \quad (4.25)$$

$$G_\phi^{-1}(x) \simeq -B^{-1} i^{1-D} 2^{2-\eta_\phi} \pi^{-\frac{D}{2}} a_D \left( \frac{-2+\eta_\phi}{2} \right) |x|^{-D-2+\eta_\phi}, \quad (4.26)$$

with  $a_D(\alpha) = \Gamma(D/2 - \alpha)/\Gamma(\alpha)$  as before.

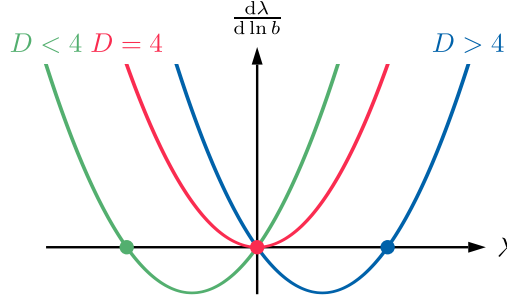
### Skeleton Dyson-Schwinger equations

The third step consists of inserting the above asymptotic expressions into the Dyson-Schwinger equations for the two-point functions in the real-space representation [95, 461, 462]

$$G_\psi^{-1}(x) = G_{\psi,0}^{-1}(x) + \Sigma_\psi(x), \quad (4.27)$$

$$G_\phi^{-1}(x) = G_{\phi,0}^{-1}(x) + \Sigma_\phi(x), \quad (4.28)$$

where  $G_{\psi,0}^{-1}$  ( $G_{\phi,0}^{-1}$ ) denotes the inverse Fourier transform of the inverse bare fermion (boson) propagator and, analogously,  $\Sigma_\psi$  ( $\Sigma_\phi$ ) represents the inverse Fourier transform of the fermion (boson) self-energy. At  $N < \infty$ , both contributions from the bare propagators,  $G_{\psi,0}^{-1}$  and  $G_{\phi,0}^{-1}$ , are subleading compared to the contributions from the interacting propagators,  $G_\psi^{-1}$  and  $G_\phi^{-1}$ , and can therefore be omitted [95]. As in the previous section, interaction effects are taken into account by means of a skeleton expansion of the self-energies [78, 95, 463–465]. To leading order in  $|x| \ll 1$ , both self-energies contribute only through the resummed one-loop diagrams depicted in Fig. 4.4. Inserting Eqs. (4.23)–(4.26) into the one-loop skeleton Dyson-Schwinger equations yields two independent equations involving the unknown  $A^2 B g^2$ . Eliminating this factor and expanding the resulting equa-



**Figure 4.5.:** Schematic representation of the one-loop beta function of the four-fermion interaction  $\lambda$ , as given in Eq. (4.30), for different space-time dimensions  $D$ . For  $D < 4$ , the beta function possesses an interacting, infrared stable fixed point at  $\lambda_* < 0$  (green dot). In  $D = 4$  space-time dimensions, the renormalization group flow undergoes a transcritical bifurcation (Fig. 1.6(b)), and the previously interacting fixed point coincides with the Gaussian fixed point at  $\lambda_* = 0$  (red dot). For  $D > 4$ , the beta function possesses an interacting, ultraviolet stable fixed point at  $\lambda_* > 0$  (blue dot). This situation occurs for any number of total fermionic components  $Nd_\gamma > 2$ .

tion to lowest nontrivial order in  $1/N$  gives the first-order contribution

$$\eta_\psi^{(1)}(D) = \frac{-32\Gamma(D-2)}{(D-4)(D-2)D(D+2)\Gamma\left(\frac{4-D}{2}\right)\Gamma^2\left(\frac{D-4}{2}\right)\Gamma\left(\frac{D}{2}\right)d_\gamma} \quad (4.29)$$

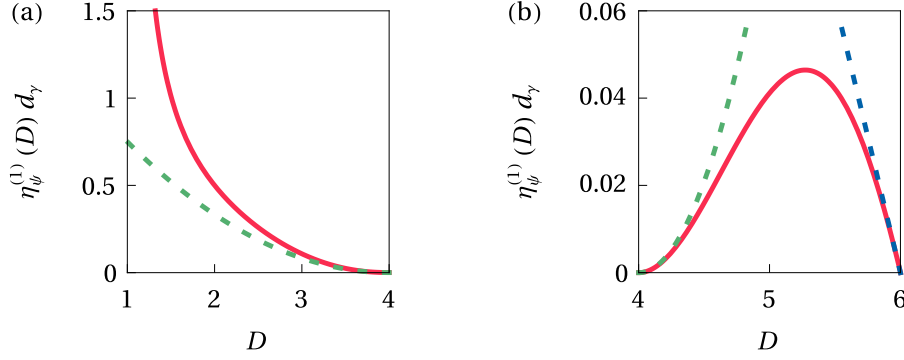
to the fermion anomalous dimension  $\eta_\psi = \sum_{n=0}^{\infty} \eta_\psi^{(n)}(D)(1/N)^n$ , cf. Fig. 4.1. As in most fermion-boson models discussed in this dissertation, the contribution from random-phase approximation vanishes,  $\eta_\psi^{(0)}(D) = 0$ , which follows from the above skeleton Dyson-Schwinger equations as a byproduct. First-order contributions to other scaling exponents, such as the boson anomalous dimension  $\eta_\phi = \sum_{n=0}^{\infty} \eta_\phi^{(n)}(D)(1/N)^n$ , are, however, more difficult to obtain. This is because they require an analysis of the subleading terms in  $|x|$ , including the incorporation of two-loop skeleton diagrams, which lies beyond the scope of our discussion here. However, the boson anomalous dimension in random-phase approximation,  $\eta_\phi^{(0)}(D) = 6 - D$ , again follows from the analysis above.

## Interpretation

In  $1 < D < 4$  space-time dimensions, Eq. (4.29) characterizes a novel fermionic conformal fixed point. Its existence can already be anticipated from the flow equation of the corresponding dimensionless four-fermion coupling  $\lambda$  at the one-loop order [336]

$$\frac{d\lambda}{d \ln b} = (4 - D)\lambda + f(Nd_\gamma, D)\lambda^2, \quad (4.30)$$

where  $\ln b$  denotes the renormalization group scale, with  $\ln b = 0$  ( $\ln b \rightarrow \infty$ ) corresponding to the ultraviolet (infrared). Since  $f(Nd_\gamma, D) := 16(Nd_\gamma - 2)/[D2^{D+1}\pi^{D/2}\Gamma(D/2)] > 0$  for any reasonable choice of the number of total fermionic components  $Nd_\gamma > 2$  [336], there exists an interacting fixed point at  $\lambda_* < 0$  below  $D = 4$  space-time dimensions. This fixed point, illustrated in Fig. 4.5, is precisely the fixed point that is characterized by the fermion anomalous dimension  $\eta_\psi \simeq \eta_\psi^{(1)}(D)/N$  as computed above. The latter is



**Figure 4.6.:** First-order contribution  $\eta_\psi^{(1)}(D)d_\gamma$  to the fermion anomalous dimension  $\eta_\psi = \sum_{n=0}^{\infty} \eta_\psi^{(n)}(D)(1/N)^n$  obtained within the relativistic large- $N$  critical point formalism. While solid red lines depict the nonperturbative result, dashed green (blue) lines visualize its asymptotic behavior near the lower (upper) critical dimension. (a) For all space-time dimensions  $1 < D < 4$ ,  $\eta_\psi^{(1)}(D)$  characterizes a novel fermionic conformal fixed point, which constitutes an infrared stable fixed point, associated with a conformal phase, for  $\frac{8}{3} < D < 4$ . The simple pole at  $D = 1$  is known from conventional Gross-Neveu-Ising theories, for instance. (b) In  $4 < D < 6$  space-time dimensions,  $\eta_\psi^{(1)}(D)$  describes an interacting ultraviolet stable fixed point, revealing a new asymptotically safe quantum field theory. Although this graph may look similar to the one presented in Fig. 4.3, we emphasize that they are not equivalent but, besides an overall factor, exhibit (relatively small) differences.

visualized in Fig. 4.6(a) for different space-time dimensions  $1 < D < 4$ .

What is the nature of this interacting fixed point? As indicated by the flow equation of the four-fermion interaction  $\lambda$ , it is an infrared stable fixed point near and below  $D = 4$  space-time dimensions. This statement remains valid in any space-time dimension  $\frac{8}{3} < D < 4$  at sufficiently large  $N$ , where  $D = \frac{8}{3}$  is the dimension at (below) which eight-fermion interactions become marginal (relevant). Although the physics below  $D = \frac{8}{3}$  is interesting in its own right, we focus in this dissertation on the infrared stable conformal fixed point in  $\frac{8}{3} < D < 4$  space-time dimensions and continue with a few comments. We note that the infrared stability of this fixed point can also be seen within a partially bosonized formulation of the theory, cf. Eq. (4.20). In particular, the partial derivative of the boson-mass flow equation with respect to the boson mass  $r$  takes the form<sup>11</sup>

$$\left. \frac{\partial}{\partial r} \frac{dr}{d \ln b} \right|_{g_\star^2, r_\star} = 2 - \eta_\phi^{(0)}(D) + \mathcal{O}\left(\frac{1}{N}\right), \quad (4.31)$$

where  $\eta_\phi^{(0)}(D) = 6 - D > 2$  (see below Eq. (4.29)) in  $1 < D < 4$  space-time dimensions. Thus, this gives a negative entry in the stability matrix. Second, we emphasize that the existence of the infrared stable fixed point does not require the presence of a gauge field. This is in contrast to all higher-dimensional infrared stable fixed points discussed in this thesis, namely the Luttinger-Abrikosov-Beneslavskii fixed point of nonrelativistic Luttinger fermion systems (see Sec. 3.5), the conformal fixed point of quantum electrody-

<sup>11</sup>We note that below  $D = 4$  space-time dimensions the quartic bosonic self-interaction  $u\phi^4$  should be incorporated in Eq. (4.20). At the interacting fixed point  $(g_\star^2, r_\star, u_\star)$ , we expect, in analogy with ordinary Gross-Neveu field theories, the one-loop contribution  $\propto u$  to Eq. (4.31) to be subleading.

namics in  $2 + 1$  space-time dimensions (see Sec. 3.6), and the Caswell-Banks-Zaks fixed point of quantum chromodynamics in  $3 + 1$  space-time dimensions (see Sec. 3.8). From this perspective, the present fixed point bears some similarity to the interacting disordered fixed point of the spin-boson model with XXZ anisotropy (and, of course, also to the infrared stable fixed point in the  $SU(2)$ -symmetric case), as encountered in Sec. 3.7. In the future, one may investigate whether the infrared stable conformal fixed point of relativistic Luttinger fermion systems can annihilate with a quantum critical fixed point by varying the flavor number  $N$  and/or the space-time dimension  $D$ . If a fixed-point annihilation indeed occurs, one may attempt to identify a continuous order-to-order quantum phase transition as outlined in Chap. 3.

Alternatively to the scenario outlined above, we may understand  $\eta_\psi \simeq \eta_\psi^{(1)}(D)/N$ , as given in Eq. (4.29), as a scaling exponent characterizing an interacting ultraviolet stable fixed point at  $\lambda_* > 0$  in  $4 < D < 6$  space-time dimensions, cf. Eq. (4.30). The interacting fixed point is again schematically depicted in Fig. 4.5. Its existence prevents the occurrence of a runaway flow towards  $\lambda \rightarrow \infty$  as  $\ln b \rightarrow 0$  for any initial four-fermion coupling  $\lambda > 0$ , i.e., renders the theory asymptotically safe in the ultraviolet [470–472]. Similar fixed points have been implicitly encountered in the  $QED_3$ -Gross-Neveu model and in quantum chromodynamics in  $3 + 1$  space-time dimensions with supplemental four-fermion interactions, as depicted in Figs. 3.11 and 3.14, respectively (see UV labels). Further, the scaling exponent given in Eq. (4.29) is visualized for  $4 < D < 6$  space-time dimensions in Fig. 4.6(b) and allows for the following consistency check. Expanding the first-order contribution to the fermion anomalous dimension around the upper and lower critical dimension yields

$$\eta_\psi^{(1)}(D) = \frac{6-D}{8d_\gamma} + \mathcal{O}\left((6-D)^2\right) \quad \text{and} \quad \eta_\psi^{(1)}(D) = \frac{(D-4)^2}{12d_\gamma} + \mathcal{O}\left((D-4)^3\right), \quad (4.32)$$

respectively, which agree with our expectations that the former possesses a nonzero linear contribution while the latter does not. Both expressions are expected to agree quantitatively with future one-loop (two-loop) results from  $6-D$  ( $D+4$ ) expansion, when expanding the latter in  $1/N$ . Further, in  $4 < D < 6$  space-time dimensions, relativistic Luttinger fermions behave somewhat similarly to linear sigma models within a Hubbard-Stratonovich formulation above four space-time dimensions [378–380]. Their shared phenomenology, including ultraviolet completion (at least at large  $N$ ), can be seen as a consequence of the coinciding canonical scaling dimensions of the fields and the (super-renormalizable) Yukawa interaction.

In summary, relativistic Luttinger fermions are indeed accessible within the relativistic large- $N$  critical point formalism and their nontrivial solution can be constructed in the standard way. Apart from this formal result, we find a novel fermionic conformal fixed point, which may be understood either as an infrared stable fixed point, associated with a conformal phase, in  $\frac{8}{3} < D < 4$  space-time dimensions or as an ultraviolet stable fixed point in  $4 < D < 6$  space-time dimensions, demonstrating once more that fermionic

degrees of freedom are able to generate both fascinating infrared and ultraviolet behavior.

## 4.4. Putative Sachdev-Ye-Kitaev model in three dimensions

In this third and final section on ongoing research, we return to the conceptual motivation presented in Sec. 4.1.2 and provide a brief outlook on a putative fixed point in three spatial dimensions with SYK-like features.

### 4.4.1. Model

In analogy with the construction of Kim et al. [449], we consider a random version of a partially bosonized Luttinger fermion model in this section. Since the large- $N$  behavior of nonrelativistic Luttinger semimetal field theories is best understood in the  $s$ -wave superconducting model defined by  $S = S_0 + S_\phi$ , cf. Eqs. (4.2)–(4.3), we take this theory as the starting point for constructing a random variant. The random analogue is characterized by the Euclidean action  $S_r = S_0 + S_{\phi,r}$ , where  $S_0$  describes  $N$  flavors of noninteracting Luttinger fermions, cf. Eq. (4.2), and

$$S_{\phi,r} = \int d\tau d^3x \left[ \sum_{k=1}^M r |\phi_k|^2 + \sum_{i,j=1}^N \sum_{k=1}^M \left( g_{ijk} \psi_i^\dagger \gamma_{45} \psi_j^* \phi_k + g_{jik}^* \psi_i^\dagger \gamma_{45} \psi_j \phi_k^* \right) \right] \quad (4.33)$$

captures the  $M$ -component  $s$ -wave superconducting order parameter  $\phi$ , where  $r$  denotes its mass and  $\gamma_{45} = i\gamma_4\gamma_5$ , as before. Instead of coupling the fermionic quasiparticles to the order parameter through a deterministic Yukawa interaction, we use random couplings  $g_{ijk}$ , which are drawn from a complex, zero-mean Gaussian distribution with variance  $\langle\langle g_{ijk} g_{lmn}^* \rangle\rangle = g^2 / (N^2 d_\gamma) (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \delta_{kn}$ , pseudo-variance  $\langle\langle g_{ijk} g_{lmn} \rangle\rangle = 0$ , and  $\langle\langle g_{ijk}^* g_{lmn}^* \rangle\rangle = 0$  for all  $i, j, l, m \in \{1, \dots, N\}$  and  $k, n \in \{1, \dots, M\}$ , where  $g^2 > 0$ .

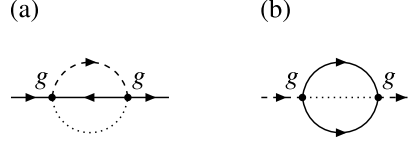
### 4.4.2. Towards a critical solution

One may analyze the random Luttinger model defined above following a similar strategy to the one employed for the nonrelativistic Luttinger model with deterministic Yukawa interaction studied in Sec. 4.2. More precisely, one can easily write down a promising ansatz for the critical ( $r = 0$ ), interacting propagators  $\tilde{G}_\psi$  and  $\tilde{G}_\phi$  in frequency and momentum space

$$\tilde{G}_\psi(\omega, \vec{k}) = \frac{A}{k^{2-\eta_\psi}} \left[ i \operatorname{sgn}(\omega) \tilde{g}_\psi^{(1)} \left( \frac{|\omega|}{k^z} \right) \mathbb{1} + \tilde{g}_\psi^{(2)} \left( \frac{|\omega|}{k^z} \right) d_a(\hat{k}) \gamma_a \right] + \dots, \quad (4.34)$$

$$\tilde{G}_\phi(\omega, \vec{k}) = \frac{B}{k^{-4+d+z+2\eta_\psi}} \tilde{g}_\phi \left( \frac{|\omega|}{k^z} \right) + \dots, \quad (4.35)$$

where  $A$  and  $B$  denote frequency- and momentum-independent amplitudes, and  $\eta_\psi$  represents the fermion anomalous dimension. In the above ansatz, we keep the flavor- and



**Figure 4.7.:** Skeleton diagrams at the one-loop order contributing to the (a) fermion and (b) boson self-energies. Solid (dashed) lines denote dressed fermion (complex boson) propagators, dots represent bare Yukawa vertices, and dotted lines indicate the corresponding disorder averages.

component-diagonal structure of the fermion and boson propagators, respectively, implicit. Again, we have to incorporate the dynamical exponent  $z$  and three dimensionless real-valued scaling functions,  $\tilde{g}_\psi^{(1)}$ ,  $\tilde{g}_\psi^{(2)}$ , and  $\tilde{g}_\phi$ , since we are dealing with a nonrelativistic problem. As in Sec. 4.2.2, summation over repeated indices  $a = 1, \dots, 5$  is assumed and  $\hat{k} := \vec{k}/k$ , with  $k := |\vec{k}|$ , such that  $d_a(\hat{k})$  become the  $\ell = 2$  real spherical harmonics,  $d_a(\hat{k}) = \sqrt{\frac{d}{2(d-1)}} \hat{k}^\top \Lambda_a \hat{k}$ , where  $\Lambda_a$  denote the real Gell-Mann matrices [331]. Further, the ellipses denote corrections-to-scaling terms. In contrast to the ansatz for the deterministic model, Eqs. (4.4)–(4.5), the boson anomalous dimension  $\eta_\phi$  is no longer an independent exponent here, but fully determined by the fermion anomalous dimension  $\eta_\psi$  and the dynamical exponent  $z$  through the relation  $\eta_\phi = 6 - d - z - 2\eta_\psi$ .

As discussed in Sec. 4.2.2, one may first simplify the above ansatz by accounting only for the asymptotic frequency behavior, and, in a second step, take the inverse Fourier transform of the simplified propagators and their inverses. Likewise, the skeleton Dyson-Schwinger equations at the one-loop order, cf. Fig. 4.7, are readily written down. In the Veneziano-like limit [421, 449], i.e., for  $M, N \rightarrow \infty$  with a fixed ratio of bosonic to total fermionic components  $\rho := M/(Nd_\gamma)$ , the fermionic and bosonic Dyson-Schwinger equations take the form

$$\mathcal{O}(1) = A^2 B g^2 \times \rho \times \mathcal{O}(1), \quad (4.36)$$

$$\mathcal{O}(1) = A^2 B g^2 \times \mathcal{O}(1), \quad (4.37)$$

respectively, with  $A^2 B g^2 = \mathcal{O}(1)$ . Analogous to Sec. 4.2.2, the above Dyson-Schwinger equations contain additional, potentially ill-defined contributions that arise from the frequency Fourier transform of the asymptotic ansatz<sup>12</sup>. Therefore, although the two sections consider different large- $N$  limits, they seem to share the same technical challenges, which may be phrased as: (1) How should the scaling functions,  $\tilde{g}_\psi^{(1)}$ ,  $\tilde{g}_\psi^{(2)}$ , and  $\tilde{g}_\phi$ , be treated? (2) Are both tensor structures in the fermionic sector equally relevant? We have put finding the full SYK solution on hold for now and currently investigate its slightly simpler, deterministic analogue as outlined in the last paragraph of Sec. 4.2.2.

<sup>12</sup>We expect analogous difficulties to arise when considering other random variants of the underlying deterministic superconducting Luttinger fermion system [473, 474].

## 4.5. Conclusion

In this chapter, we have presented multiple lines of ongoing research, all working towards the identification of a putative SYK fixed point in three spatial dimensions. In particular, we have analyzed a nonrelativistic Luttinger fermion model at a large but finite number of quadratic band touching points  $N$ . At the current level, our analysis is able to correctly reproduce the large- $N$  limit and a preliminary expression for the  $1/N$  contribution to the fermion anomalous dimension. However, the structure of the skeleton Dyson-Schwinger equations remains only partially understood. In a similar way, a random version of the deterministic nonrelativistic Luttinger fermion model has been analyzed, although a full solution is again currently absent. These complications, which ultimately arise from the absence of Lorentz symmetry, have motivated the study of a slightly simpler analogue, a relativistic Luttinger fermion theory, with the hope of gaining a deeper understanding of Luttinger systems at large  $N$  in general. This study has not only led to an improved understanding, but has also revealed a novel fermionic conformal fixed point at large  $N$ , which corresponds to an infrared stable fixed point in  $\frac{8}{3} < D < 4$  space-time dimensions and to an ultraviolet stable fixed point in  $4 < D < 6$  space-time dimensions.

The previously unknown interacting conformal fixed point of relativistic Luttinger fermion systems is an excellent starting point for future theoretical work, both from the viewpoint of conformal phases in condensed matter physics and from the perspective of asymptotic safety in high-energy physics [470–472]. Below four space-time dimensions, one may, for instance, investigate whether the infrared stable conformal fixed point can annihilate with a quantum critical fixed point by varying the flavor number  $N$  and/or the space-time dimension  $D$ . If a fixed-point annihilation indeed occurs, one may subsequently attempt to identify a continuous order-to-order quantum phase transition as outlined in Chap. 3. Further, one may try to construct a lattice model that realizes the conformal phase described by the infrared stable fixed point, or, above four space-time dimensions, investigate the stability of relativistic Luttinger fermion systems [475].

Further, the exotic conformal fixed point of relativistic Luttinger fermion systems motivates the search for an analogous fixed point in relativistic Luttinger fermion systems with random, rather than deterministic, Yukawa interactions. In particular, one might ask whether such an SYK fixed point is again infrared stable in  $D = 3$  space-time dimensions, which would set it apart from the quantum critical fixed point found in the random Gross-Neveu model in  $D = 3$  [449]. In fact, we have recently pursued precisely this research direction and our preliminary results suggest the existence of interacting conformal fixed points in  $D = 3, 4, 5$  space-time dimensions in the limit of infinitely many bosonic and fermionic degrees of freedom,  $M, N \rightarrow \infty$  [449]. Their interpretation is currently under investigation<sup>13</sup>. Apart from revealing novel strongly correlated physics, these fixed points

<sup>13</sup>We note that the existence of scale-invariant solutions in  $D = 4, 5$  space-time dimensions already seems to set our findings apart from those presented in Ref. [449].

may also help gain a better understanding of nonrelativistic Luttinger fermion systems with random Yukawa interactions, cf. Sec. 4.4.

As outlined in detail in the last paragraph of Sec. 4.2.2, another promising direction for future research is the investigation of nonrelativistic Luttinger fermions within a generalization of the large- $N$  framework developed in the context of  $m$ -axial Lifshitz points in  $d$  spatial dimensions [431, 432]. Such a generalization is currently being pursued and appears promising with respect to an appropriate incorporation of the scaling functions. In the case of a positive outcome, we also expect progress towards identifying a large- $N$  SYK-like solution in three spatial dimensions, since deterministic nonrelativistic Luttinger fermions behave very similarly to their randomly coupled counterparts on a technical level, cf. Sec. 4.4.2. Interesting follow-up questions are manifold, ranging from investigating non-relativistic holographic settings [476] to analyzing out-of-time-order correlations [449] and exploring the onset of superconductivity [454, 455].

## 5. Concluding remarks

*“Most people who haven’t been trained in physics probably think of what physicists do as a question of incredibly complicated calculations, but that’s not really the essence of it. The essence of it is that physics is about concepts, wanting to understand the concepts, the principles by which the world works.”* [1]

— Edward Witten

We would like to end this dissertation as we began, by reflecting on the different principles encountered in the field of phase transitions. In Chap. 1, we have reviewed the Landau-Ginzburg-Wilson paradigm and explored various routes beyond it. In particular, we have examined phase transitions beyond conventional symmetry breaking, such as the Berezinskii-Kosterlitz-Thouless transition or transitions between phases with distinct topological order, as well as phase transitions enriched by additional gapless modes, such as fermionic phase transitions, fluctuation-induced first-order phase transitions, or deconfined phase transitions. A visual overview of these routes beyond the Landau-Ginzburg-Wilson paradigm, and of this dissertation in general, is given in Fig. 1.10.

In Chap. 2, the first of three research chapters, we have proposed a novel mechanism for quasiuniversality that relies on the presence of a marginally irrelevant operator at the quantum critical fixed point. In contrast with fixed-point annihilation scenarios [271, 274, 275], the proposed mechanism offers a unique scenario in which quasiuniversal behavior at intermediate-to-low-energy scales and genuine universal behavior at the ultra-low-energy scale are not contradictory, but rather both consequences of an exotic quantum critical fixed point. Within an exact renormalization group analysis, we have demonstrated the emergence of such a novel quasiuniversal regime in the finite-temperature phase diagram of interacting three-dimensional Luttinger semimetals. Experimental signatures have been proposed and align with recent experimental findings in  $\text{Y}_2\text{Ir}_2\text{O}_7$  thin films [346].

In Chap. 3, we have introduced a novel mechanism for continuous order-to-order phase transitions that relies on the collision and annihilation of two renormalization group fixed points, one infrared stable and one critical. The proposed mechanism operates independently of fractionalization, requires no fine tuning, and, in contrast to weakly first-order scenarios [153, 271], leads to genuine critical behavior. We have demonstrated its relevance across a wide range of systems, including three-dimensional Luttinger semimetals, kagome

quantum magnets, quantum impurity models, and four-dimensional quantum chromodynamics with supplemental four-fermion interactions. A distinctive experimental or numerical signature of the proposed mechanism, a pronounced asymmetry in the effective energy scales between the two sides of the transition, has been identified and manifests itself in recent quantum Monte Carlo simulations [290]. On the experimental side, the proposed mechanism appears to be relevant for several classes of materials, including the rare-earth pyrochlore iridates  $R_2\text{Ir}_2\text{O}_7$ , herbertsmithite, and kapellasite.

In Chap. 4, we have presented three strands of ongoing research, all ultimately working towards a putative Sachdev-Ye-Kitaev fixed point in three spatial dimensions. As a starting point, we have investigated the large- $N$  behavior of a partially bosonized Luttinger fermion theory at a large but finite number of quadratic band touching points  $N$ . In a technically similar way, we have analyzed a random variant of the deterministic Luttinger fermion model, now in the limit of infinitely many fermionic and bosonic degrees of freedom, as known from low-dimensional Yukawa-Sachdev-Ye-Kitaev theories [449]. In both cases, a partial solution has been obtained, which motivates continuing efforts towards a putative fixed point in three spatial dimensions with Sachdev-Ye-Kitaev-like properties. The encountered complications, which ultimately arise from the absence of Lorentz symmetry, have motivated the study of a slightly simpler analogue, a relativistic Luttinger fermion theory [336]. This third study has not only led to an improved understanding of Luttinger fermion systems at large  $N$  in general, but has also revealed a novel fermionic conformal fixed point, which corresponds to an infrared stable fixed point in  $\frac{8}{3} < D < 4$  space-time dimensions and to an ultraviolet stable fixed point in  $4 < D < 6$  space-time dimensions. While the former gives rise to a previously unknown conformal phase, the latter unveils a new asymptotically safe quantum field theory.

With a novel mechanism for quasiuniversality, a new class of continuous order-to-order phase transitions, and a first step towards a three-dimensional Sachdev-Ye-Kitaev model, we have contributed to the theory of phase transitions, in particular those beyond the Landau-Ginzburg-Wilson paradigm. In Secs. 2.7, 3.9, and 4.5, at the end of each research chapter, we have provided numerous detailed proposals for future analytical, numerical, and experimental investigations. In the spirit of Witten's quote above, these suggestions may help to build on the concepts introduced in this dissertation and, in turn, further advance our understanding of phase transitions.

## A. Solid-angle integrals

In this appendix, we provide definitions and numerical values of the solid-angle integrals  $f_i \equiv f_i(\delta)$ , with  $i \in \{1, 1t, 1e, 2, 2t, 2e, 3, 3t, 3e, 3\bar{t}, 4, 4t, 4e, 4tt, 4tt', 4ee, 4et, 4t\bar{t}, 4e\bar{t}, e^2, g^2\}$ , as functions of the anisotropy parameter  $\delta$ , occurring in the mean-field calculations, Sec. 2.3 and Sec. 3.4, and the renormalization group analyses, Sec. 2.4 and Sec. 3.5. These are bounded and continuous functions of order unity with  $f_i > 0$  ( $f_{g^2} \geq -2/3$ ) for  $\delta \in [-1, 1]$  and  $f_i = 1$  ( $f_{g^2} = 0$ ) for  $\delta = 0$ . Some of these have already been defined in Ref. [327], reading

$$f_1(\delta) := \frac{1}{4\pi} \int d\Omega \frac{1}{\tilde{X}^{1/2}}, \quad (\text{A.1})$$

$$f_{1t}(\delta) := \frac{5}{4\pi} \int d\Omega \frac{\tilde{d}_1^2}{\tilde{X}^{1/2}}, \quad (\text{A.2})$$

$$f_{1e}(\delta) := \frac{5}{4\pi} \int d\Omega \frac{\tilde{d}_4^2}{\tilde{X}^{1/2}}, \quad (\text{A.3})$$

$$f_2(\delta) := \frac{1}{4\pi} (1 - \delta)(1 + \delta) \int d\Omega \frac{1}{\tilde{X}^{3/2}}, \quad (\text{A.4})$$

$$f_{2t}(\delta) := \frac{5}{4\pi} (1 + \delta) \int d\Omega \frac{\tilde{d}_1^2}{\tilde{X}^{3/2}}, \quad (\text{A.5})$$

$$f_{2e}(\delta) := \frac{5}{4\pi} (1 - \delta) \int d\Omega \frac{\tilde{d}_4^2}{\tilde{X}^{3/2}}, \quad (\text{A.6})$$

$$f_3(\delta) := \frac{1}{4\pi} (1 - \delta)^3 (1 + \delta)^3 \int d\Omega \frac{1}{\tilde{X}^{5/2}}, \quad (\text{A.7})$$

$$f_{3t}(\delta) := \frac{5}{4\pi} (1 - \delta)(1 + \delta)^3 \int d\Omega \frac{\tilde{d}_1^2}{\tilde{X}^{5/2}}, \quad (\text{A.8})$$

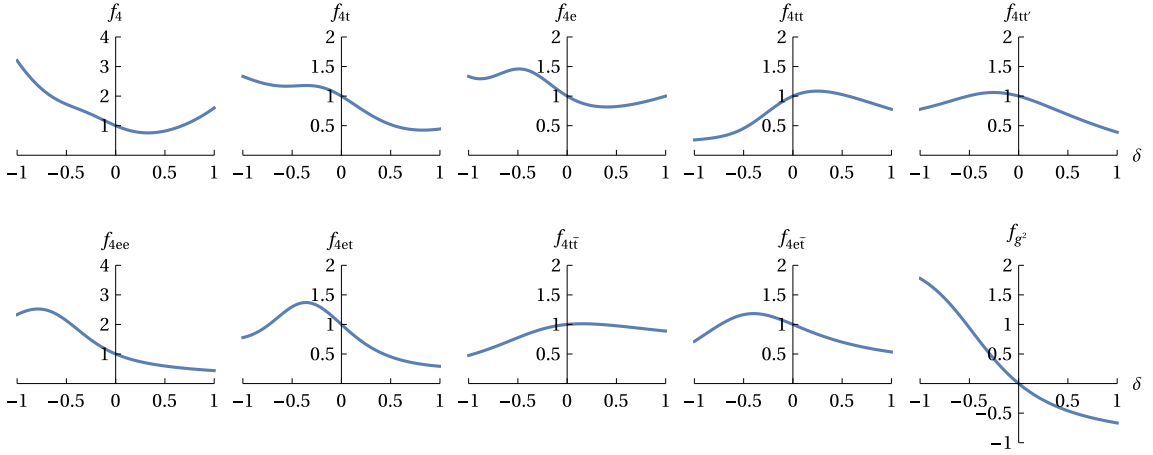
$$f_{3e}(\delta) := \frac{5}{4\pi} (1 - \delta)^3 (1 + \delta) \int d\Omega \frac{\tilde{d}_4^2}{\tilde{X}^{5/2}}, \quad (\text{A.9})$$

$$f_{3\bar{t}}(\delta) := \frac{35}{\sqrt{34}\pi} (1 + \delta)^3 \int d\Omega \frac{\tilde{d}_1 \tilde{d}_2 \tilde{d}_3}{\tilde{X}^{5/2}}, \quad (\text{A.10})$$

where  $\int d\Omega := \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi$  denotes the integration over the solid angle,  $\tilde{X}(\theta, \phi) := (1 - \delta)^2 + 12\delta \sum_{i < j} \frac{q_i^2 q_j^2}{q^2}$ , and  $\tilde{d}_a(\theta, \phi) := d_a(\vec{q})/q^2$  are the  $\ell = 2$  real spherical harmonics. Upon spatial rotations, the latter transform under the irreducible representation  $T_{2g}$  ( $E_g$ ) of the octahedral point group  $O_h$  for  $a = 1, 2, 3$  ( $a = 4, 5$ ), and the indices t and e of the functions  $f_i$  indicate the type of spherical harmonics involved in the integral. In addition to the above-defined functions, the loop expansion of the AIAO order-parameter field

**Table A.1.:** Values of solid-angle integrals  $f_i(\delta)$  for limiting cases  $\delta = \pm 1$  and isotropic case  $\delta = 0$  from, whenever possible, analytical integration, otherwise determined numerically.

| $f_i$    | $f_i(-1)$ | $f_i(0)$ | $f_i(+1)$ | $f_i$          | $f_i(-1)$ | $f_i(0)$ | $f_i(+1)$ | $f_i$           | $f_i(-1)$ | $f_i(0)$ | $f_i(+1)$ |
|----------|-----------|----------|-----------|----------------|-----------|----------|-----------|-----------------|-----------|----------|-----------|
| $f_1$    | 1.0942    | 1        | 0.8130    | $f_{3t}$       | 5/9       | 1        | 5/9       | $f_{4tt'}$      | 7/9       | 1        | 7/18      |
| $f_{1t}$ | 1.3294    | 1        | 0.6225    | $f_{3e}$       | 5/3       | 1        | 5/12      | $f_{4ee}$       | 7/3       | 1        | 7/16      |
| $f_{1e}$ | 0.7415    | 1        | 1.0987    | $f_{3\bar{t}}$ | 35/54     | 1        | 0.7262    | $f_{4et}$       | 7/9       | 1        | 7/24      |
| $f_2$    | 1         | 1        | 1/2       | $f_4$          | 16/5      | 1        | 8/5       | $f_{4t\bar{t}}$ | 77/162    | 1        | 0.8876    |
| $f_{2t}$ | 5/6       | 1        | 0.6775    | $f_{4t}$       | 4/3       | 1        | 4/9       | $f_{4e\bar{t}}$ | 77/108    | 1        | 77/144    |
| $f_{2e}$ | 1.3678    | 1        | 5/8       | $f_{4e}$       | 4/3       | 1        | 1         | $f_{e^2}$       | 16/9      | 1        | 4/3       |
| $f_3$    | 4/3       | 1        | 2/3       | $f_{4tt}$      | 7/27      | 1        | 7/9       | $f_{g^2}$       | 16/9      | 0        | -2/3      |



**Figure A.1.:** Graphs of solid-angle integrals  $f_i$  as functions of  $\delta$ . Except for  $f_{g^2}$ , all solid-angle integrals satisfy  $f_i > 0$  for  $\delta \in [-1, 1]$  and  $f_i = 1$  for  $\delta = 0$ . Graphs not shown here are given in Ref. [327]. Figure reprinted from Ref. [301].

theory presented in Sec. 2.4 gives rise to the following new solid-angle integrals,

$$f_4(\delta) := \frac{1}{4\pi} (1-\delta)^5 (1+\delta)^5 \int d\Omega \frac{1}{\tilde{X}^{7/2}}, \quad (\text{A.11})$$

$$f_{4t}(\delta) := \frac{5}{4\pi} (1-\delta)^3 (1+\delta)^5 \int d\Omega \frac{\tilde{d}_1^2}{\tilde{X}^{7/2}}, \quad (\text{A.12})$$

$$f_{4e}(\delta) := \frac{5}{4\pi} (1-\delta)^5 (1+\delta)^3 \int d\Omega \frac{\tilde{d}_4^2}{\tilde{X}^{7/2}}, \quad (\text{A.13})$$

$$f_{4tt}(\delta) := \frac{35}{12\pi} (1-\delta)(1+\delta)^5 \int d\Omega \frac{\tilde{d}_1^2 \cdot \tilde{d}_1^2}{\tilde{X}^{7/2}}, \quad (\text{A.14})$$

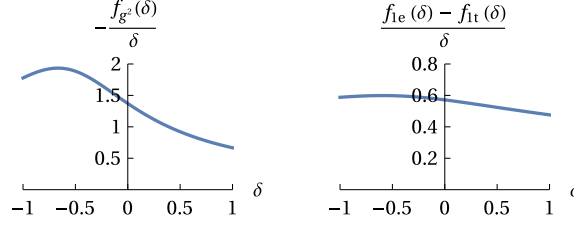
$$f_{4tt'}(\delta) := \frac{35}{4\pi} (1-\delta)(1+\delta)^5 \int d\Omega \frac{\tilde{d}_1^2 \cdot \tilde{d}_2^2}{\tilde{X}^{7/2}}, \quad (\text{A.15})$$

$$f_{4ee}(\delta) := \frac{35}{12\pi} (1-\delta)^5 (1+\delta) \int d\Omega \frac{\tilde{d}_4^2 \cdot \tilde{d}_4^2}{\tilde{X}^{7/2}}, \quad (\text{A.16})$$

$$f_{4et}(\delta) := \frac{35}{4\pi} (1-\delta)^3 (1+\delta)^3 \int d\Omega \frac{\tilde{d}_1^2 \cdot \tilde{d}_4^2}{\tilde{X}^{7/2}}, \quad (\text{A.17})$$

$$f_{4t\bar{t}}(\delta) := \frac{385}{\sqrt{3}12\pi} (1+\delta)^5 \int d\Omega \frac{\tilde{d}_1^2 \cdot \tilde{d}_1 \tilde{d}_2 \tilde{d}_3}{\tilde{X}^{7/2}}, \quad (\text{A.18})$$

$$f_{4e\bar{t}}(\delta) := \frac{385}{\sqrt{3}4\pi} (1-\delta)(1+\delta)^3 \int d\Omega \frac{\tilde{d}_4^2 \cdot \tilde{d}_1 \tilde{d}_2 \tilde{d}_3}{\tilde{X}^{7/2}}. \quad (\text{A.19})$$



**Figure A.2.:** Graphs of two frequently encountered expressions,  $-\frac{f_{g^2}(\delta)}{\delta}$  (left) and  $\frac{f_{1e}(\delta) - f_{1t}(\delta)}{\delta}$  (right), arising in the renormalization group analysis presented in Sec. 3.5, as functions of anisotropy parameter  $\delta$ . Most importantly, both functions only possess a removable singularity at  $\delta = 0$ . Figure reprinted from Ref. [283].

For the Coulomb and AIAO-order-parameter anomalous dimensions, it is convenient to define two further functions as combinations of the above-defined ones,

$$f_{e^2}(\delta) := \frac{1}{3} \left[ 2(1 - \delta)^2 + 3(1 + \delta)^2 \right] f_2(\delta) - \frac{2}{3} \left[ \frac{2}{5}(1 - \delta)^2 f_{3e}(\delta) + \frac{3}{5}(1 + \delta)^2 f_{3t}(\delta) - \frac{12}{5} \frac{\delta^2}{(1 + \delta)^2} f_{3t}(\delta) + \frac{36}{35} \frac{\delta^2(1 - \delta)}{(1 + \delta)^2} f_{3\bar{t}}(\delta) \right], \quad (\text{A.20})$$

$$f_{g^2}(\delta) := \frac{2(1 - \delta)^2 - (1 + \delta)^2}{4} f_2(\delta) + \frac{8\delta^2 - 6(1 - \delta)^2(1 + \delta)^2 - 11(1 + \delta)^4}{20(1 + \delta)^2} f_{3t}(\delta) + \frac{4(1 - \delta)^2 + 9(1 + \delta)^2}{30} f_{3e}(\delta) - \frac{6\delta^2(1 - \delta)}{35(1 + \delta)^2} f_{3\bar{t}}(\delta) + \left\{ \frac{(1 - \delta)^2 [-4(1 - \delta)^2 + 7(1 + \delta)^2]}{14(1 + \delta)^2} + \frac{2\delta^2 [-(1 - \delta)^2 + 9(1 + \delta)^2]}{7(1 + \delta)^4} \right\} f_{4tt}(\delta) + \left\{ \frac{(1 - \delta)^2 [-5(1 - \delta)^2 + 11(1 + \delta)^2]}{42(1 + \delta)^2} + \frac{2\delta^2 [(1 - \delta)^2 + 18(1 + \delta)^2]}{21(1 + \delta)^4} \right\} f_{4tt'}(\delta) - \frac{4(1 - \delta)^2}{21} f_{4ee}(\delta) + \frac{54\delta^2(1 - \delta)}{77(1 + \delta)^2} f_{4tt\bar{t}}(\delta) - \frac{4\delta [(1 - \delta)^2 + 3(1 + \delta)^2]}{21(1 + \delta)^2} f_{4et}(\delta) - \frac{12\delta^2(1 - \delta)^2}{77(1 + \delta)^2} f_{4e\bar{t}}(\delta). \quad (\text{A.21})$$

We emphasize that all  $f_i$  are bounded from above and below for all  $\delta \in [-1, 1]$ ; in particular, they remain finite in the limiting cases  $\delta = \pm 1$ , see Table A.1. Figure A.1 shows the graphs of those functions  $f_i$  that have not already been visualized in Ref. [327].

Further, the two ratios,  $-\frac{f_{g^2}(\delta)}{\delta}$  and  $\frac{f_{1e}(\delta) - f_{1t}(\delta)}{\delta}$ , arise frequently in the renormalization

group analysis presented in Sec. 3.5. Their graphs are displayed in Fig. A.2. Most importantly, both ratios are well defined in the isotropic limit  $\delta \rightarrow 0$ , namely  $\lim_{\delta \rightarrow 0} \frac{f_{g^2}(\delta)}{-\delta} = \frac{48}{35}$  and  $\lim_{\delta \rightarrow 0} \frac{f_{1e}(\delta) - f_{1t}(\delta)}{\delta} = \frac{4}{7}$ .

## B. Master formulae for Fourier analysis

### B.1. Nonrelativistic case

In this appendix, we state three master formulae used to derive the Fourier transforms of (asymptotic) propagators and their inverses in  $(d+1)$ -dimensional Euclidean space-time, cf. Sec. 4.2.2. As in the subsection of the main text, we here employ the convention  $\mathcal{F}^{-1}[\cdot] = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}\vec{x}} \int \frac{d\omega}{2\pi} e^{i\omega\tau}$  for the inverse Fourier transform with respect to both frequency and momentum, and obtain

$$\mathcal{F}^{-1} \left[ k^\alpha \left| \frac{\omega}{k^z} \right|^\beta \right] = \frac{2^{\alpha+(1-z)\beta} a_d \left( \frac{-\alpha+z\beta}{2} \right) a_1 \left( -\frac{\beta}{2} \right)}{\pi^{\frac{d+1}{2}}} \frac{1}{x^{d+z+\alpha}} \left| \frac{\tau}{x^z} \right|^{-1-\beta}, \quad (\text{B.1})$$

$$\begin{aligned} \mathcal{F}^{-1} \left[ k^\alpha \left| \frac{\omega}{k^z} \right|^\beta d_a(\hat{k}) \gamma_a \right] &= \frac{2^{\alpha+(1-z)\beta} (-\alpha+z\beta-d) a_d \left( \frac{-\alpha+z\beta}{2} \right) a_1 \left( -\frac{\beta}{2} \right)}{\pi^{\frac{d+1}{2}} (-\alpha+z\beta)} \\ &\times \frac{1}{x^{d+z+\alpha}} \left| \frac{\tau}{x^z} \right|^{-1-\beta} d_a(\hat{x}) \gamma_a, \end{aligned} \quad (\text{B.2})$$

where  $k = |\vec{k}|$ ,  $\hat{k} = \vec{k}/k$ ,  $d_a(\hat{k}) = \sqrt{\frac{d}{2(d-1)}} \hat{k}^\top \Lambda_a \hat{k}$ ,  $\Lambda_a$  denote the real Gell-Mann matrices,  $\gamma_a$  represent the Euclidean Dirac matrices, and  $a_n(\alpha) = \Gamma(n/2 - \alpha)/\Gamma(\alpha)$  with  $n = 1, d$  as before. There exists a related formula, which is not employed in the analysis presented in the main text but proves useful in dealing with the Berry-phase contribution in the fermionic sector, reading

$$\begin{aligned} \mathcal{F}^{-1} \left[ k^\alpha \left| \frac{\omega}{k^z} \right|^\beta i \operatorname{sgn}(\omega) \mathbb{1} \right] &= - \frac{2^{\alpha+(1-z)\beta-1} \beta a_d \left( \frac{-\alpha+z\beta}{2} \right) a_1 \left( \frac{1-\beta}{2} \right)}{\pi^{\frac{d+1}{2}}} \\ &\times \frac{1}{x^{d+z+\alpha}} \left| \frac{\tau}{x^z} \right|^{-1-\beta} \operatorname{sgn}(\tau) \mathbb{1}. \end{aligned} \quad (\text{B.3})$$

All three formulae above are readily derived from the  $n$ -dimensional inverse Fourier transform [477]

$$\int \frac{d^n k}{(2\pi)^n} e^{i\vec{k}\vec{x}} k^\alpha = \frac{2^\alpha a_n \left( -\frac{\alpha}{2} \right)}{\pi^{\frac{n}{2}} x^{n+\alpha}}, \quad (\text{B.4})$$

for  $n = 1, d$ .

## B.2. Relativistic case

In this appendix, we state two master formulae used to derive the Fourier transforms of propagators and their inverses in  $D$ -dimensional Minkowski space-time with signature  $(+, -, -, \dots)$ , cf. Sec. 4.3.2. As in the subsection of the main text, we here employ the convention  $\mathcal{F}^{-1}[\cdot] = \int \frac{d^D k}{(2\pi)^D} e^{ik_\mu x^\mu}$  for the inverse Fourier transform and obtain

$$\mathcal{F}^{-1}[|k|^\alpha] = -i^{1-D} 2^\alpha \pi^{-\frac{D}{2}} a_D \left(-\frac{\alpha}{2}\right) |x|^{-D-\alpha}, \quad (\text{B.5})$$

$$\begin{aligned} \mathcal{F}^{-1}[|k|^\alpha G_{\mu\nu} \hat{k}^\mu \hat{k}^\nu] &= -i^{1-D} 2^{\alpha-2} \pi^{-\frac{D}{2}} (-D - \alpha + 2) (D + \alpha) \\ &\quad \times a_D \left(-\frac{\alpha-2}{2}\right) G_{\mu\nu} \hat{x}^\mu \hat{x}^\nu |x|^{-D-\alpha}, \end{aligned} \quad (\text{B.6})$$

where we have used the fact that the matrices  $G_{\mu\nu}$  are Lorentz-traceless (see Sec. 4.3.1 for more information),  $\hat{k} = k/|k|$ , and  $a_D(\alpha) = \Gamma(D/2 - \alpha)/\Gamma(\alpha)$  as before. Both formulae are readily derived from Eq. (B.4), bearing in mind the slightly different notation used in the previous section.

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# List of publications

Parts of the work presented in this dissertation have been published in

1. **D. J. Moser** and L. Janssen, *Quasiuniversality from all-in–all-out Weyl quantum criticality in pyrochlore iridates*, [Phys. Rev. B \*\*109\*\*, L081111 \(2024\)](#), presented in Chap. 2
2. **D. J. Moser** and L. Janssen, *Continuous order-to-order quantum phase transitions from fixed-point annihilation*, [Rep. Prog. Phys. \*\*88\*\*, 098001 \(2025\)](#), presented in Chap. 3

and contain textual overlap with the corresponding publications. Chapter 4 contains unpublished work on the critical point formalism for several Luttinger fermion systems, which we intend to publish as

3. **D. J. Moser** and L. Janssen, *Conformal phase and asymptotic safety in relativistic Luttinger fermion systems*, in preparation, presented in Sec. 4.3
4. **D. J. Moser** and L. Janssen, *Emergent scale invariance of a Sachdev-Ye-Kitaev model in three spatial dimensions*, work in progress, presented in Secs. 4.2 and 4.4



# Confirmation

I herewith declare that I have produced this paper without the prohibited assistance of third parties and without making use of aids other than those specified; notions taken over directly or indirectly from other sources have been identified as such. This paper has not previously been presented in identical or similar form to any other German or foreign examination board. In the preparation of this dissertation, artificial-intelligence-based writing tools, such as ChatGPT and DeepL, have been used under the author's direct supervision to improve language, including grammar, vocabulary, and sentence structure. This dissertation has been prepared at the Institute of Theoretical Physics at TU Dresden under the scientific supervision of Dr. Lukas Janssen and Prof. Dr. Matthias Vojta.