

Problem set 3

to be discussed on Thursday, November 26th, 2009

see <http://tu-dresden.de/physik/tqo/lehre>

Generation of Lévy-stable random numbers

Let γ be uniform on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and let W be exponential with mean 1. Assume γ and W independent. Show that

$$X = \frac{\sin \alpha \gamma}{(\cos \gamma)^{1/\alpha}} \left(\frac{\cos((1-\alpha)\gamma)}{W} \right)^{(1-\alpha)/\alpha}$$

is a Lévy-stable random variable with $\langle e^{ikX} \rangle = e^{-|k|^\alpha}$.

Gaussian Markov processes

Let $Y(t)$ be a Gaussian stochastic process with autocorrelation $\kappa(t_1, t_2)$ and variance $\sigma^2(t)$. Let $\rho(t_1, t_2) = \kappa(t_1, t_2)/(\sigma(t_1)\sigma(t_2))$. Show that the relation

$$\rho(t_3, t_1) = \rho(t_3, t_2)\rho(t_2, t_1) \quad (t_1 \leq t_2 \leq t_3)$$

is necessary and sufficient for $Y(t)$ to be a Markov process. Verify that this relation is obeyed by the Wiener process.

Ornstein-Uhlenbeck process

If $Y(t)$ is the (standard) Ornstein-Uhlenbeck process and $t > t_1 > t_2 > \dots > t_n$ show that

$$\frac{d}{dt} \langle Y(t)Y(t_1)Y(t_2) \dots Y(t_n) \rangle = -\langle Y(t)Y(t_1)Y(t_2) \dots Y(t_n) \rangle.$$

Hence if $\Phi(t, [Y])$ is a functional depending on t and all values of Y previous to t one has

$$\frac{d}{dt} \langle Y(t)\Phi(t, [Y]) \rangle = \left\langle Y(t) \frac{\partial}{\partial t} \Phi \right\rangle - \langle Y(t)\Phi \rangle.$$